

# STRONG EXISTENCE AND HIGHER ORDER FRÉCHET DIFFERENTIABILITY OF STOCHASTIC FLOWS OF FRACTIONAL BROWNIAN MOTION DRIVEN SDE'S WITH SINGULAR DRIFT

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**ABSTRACT.** In this paper we present a new method for the construction of strong solutions of SDE's with merely integrable drift coefficients driven by a multidimensional fractional Brownian motion with Hurst parameter  $H < \frac{1}{2}$ . Furthermore, we prove the rather surprising result of the higher order Fréchet differentiability of stochastic flows of such SDE's in the case of a small Hurst parameter. In establishing these results we use techniques from Malliavin calculus combined with new ideas based on a "local time variational calculus". We expect that our general approach can be also applied to the study of certain types of stochastic partial differential equations as e.g. stochastic conservation laws driven by rough paths.

## 1. INTRODUCTION

Consider a fractional Brownian motion  $B_t^H$ ,  $t \geq 0$  with Hurst parameter  $H \in (0, 1)$  on a probability space  $(\Omega, \mathfrak{A}, P)$ , that is a centered Gaussian process with a covariance structure  $R_H(t, s)$  given by

$$R_H(t, s) = E[B_t^H B_s^H] = \frac{1}{2} \left( s^{2H} + t^{2H} - |t - s|^{2H} \right)$$

for all  $t, s \geq 0$ . The fractional Brownian motion, which is a Brownian motion in the case  $H = \frac{1}{2}$ , enjoys the property of self-similarity, that is

$$\{B_{\alpha t}^H\}_{t \geq 0} \stackrel{law}{=} \{\alpha^H B_t^H\}_{t \geq 0}$$

for all  $\alpha > 0$ . In fact the fractional Brownian motion, which has a version with  $H - \varepsilon$ -Hölder continuous paths for every  $\varepsilon \in (0, H)$ , is the only stationary Gaussian process satisfying the latter property. On the other hand this process is neither a Markov process nor a (weak) semimartingale and it is a very irregular process in the sense of rough paths for small Hurst parameters. See e.g. [30] and the references therein for more information about fractional Brownian motion.

In this article we aim at analysing solutions  $X^x$  of the stochastic differential equation (SDE)

$$X_t^x = x + \int_0^t b(s, X_s^x) ds + B_t^H, \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^d, \quad (1.1)$$

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*Date:* November 10, 2015.

*2010 Mathematics Subject Classification.* 60H10, 49N60.

*Key words and phrases.* SDEs, Compactness criterion, irregular drift, Malliavin calculus, stochastic flows, Sobolev derivative.

Thank you very much.

where  $B^H$  is a  $d$ -dimensional fractional Brownian motion, whose components are one-dimensional independent fractional Brownian motions as defined above, with Hurst parameter  $H \in (0, \frac{1}{2})$  with respect to a  $P$ -augmented filtration  $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$  generated by  $B^H$  and where  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a Borel-measurable function.

If we impose a global Lipschitz and a linear growth condition uniformly in time on the drift coefficient  $b$  in (1.1), we can use the Picard iteration scheme to obtain a unique global strong solution to the SDE (1.1), that is a  $\mathcal{F}$ -adapted solution  $X_t^x$  to (1.1), which is a measurable  $L^2(\Omega)$ -functional of the driving noise.

However, a variety of important applications of such SDE's to stochastic control theory (in the case of  $H = \frac{1}{2}$ ) (see [16]) or to the statistical mechanics of infinite particle systems (see [17]) show that the use of SDE's with regular coefficients in the sense of Lipschitzianity as models for random phenomena is not suitable and that one is forced to study such equations with coefficients which are irregular, that is discontinuous or merely measurable.

One objective of our paper is the construction of unique strong solutions to the SDE (1.1) driven by rough paths in the case of multidimensional fractional noise  $B^H$  for Hurst parameters  $H < \frac{1}{2}$  and drift coefficients

$$b \in L_{\infty}^{1,\infty} := L^{\infty}([0, T]; L^1(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d)).$$

In proving this new result, we employ tools from Malliavin Calculus and local time techniques.

The analysis of strong solutions to (1.1) has been a very active field of research in various branches of mathematics over the last decades. A foundational result in this direction of research was first obtained by Zvonkin in the beginning of the 1970ties [38], who showed the existence of a unique strong solution of one-dimensional Brownian motion driven SDE's (1.1), when the drift coefficient  $b$  is merely bounded and measurable. A few years later on, the latter result was generalised by Veretennikov [35] to the multidimensional case.

More recently, Krylov and Röckner [17] gave the construction of unique strong solutions to (1.1) under integrability conditions on the (time-inhomogeneous) drift coefficient  $b$ . See also the articles [13] or [12]. In this context, we shall also mention the generalization of Zvonkin's result to the case of stochastic evolution equations in Hilbert spaces with bounded and measurable drift coefficients [4], where the authors use solutions to infinite-dimensional Kolmogorov equations to recast the singular drift term of the evolution equation in terms of a more regular expression ("Itô-Tanaka-Zvonkin trick").

In all of the above mentioned works the common technique of the authors for the construction of strong solutions rests on the so-called Yamada-Watanabe principle (see [37]), which entails strong uniqueness of solutions to SDE's, if pathwise uniqueness of (weak) solutions holds.

In fact, in order to ensure strong uniqueness of solutions, the above authors construct weak solutions to SDE's, which are not necessarily Brownian functionals, by means e.g. of [12],[13] Skorokhod embedding combined with Krylov's estimates and verify pathwise uniqueness by using solutions of parabolic partial differential equations (see e.g. [38], [35] or [17]).

We remark that the techniques of these authors for proving pathwise uniqueness are not applicable to SDE's driven by fractional Brownian motion, since the fractional Brownian is neither a Markov process nor a semimartingale for Hurst parameters  $H \neq \frac{1}{2}$ .

Further, we emphasise that our method, which is not only limited to Markov or semimartingale solutions of SDE's, gives a direct construction of strong solutions and provides a construction principle, which can be considered the converse to that of Yamada-Watanabe: We prove the existence of strong solutions and uniqueness in law to guarantee strong uniqueness.

To the best of our knowledge strong solutions to SDE's (1.1) for Hurst parameters  $H < \frac{1}{2}$  and dimension  $d \geq 2$  with irregular drift coefficients are for the first time obtained in this paper.

The case  $d = 1$  for Hurst parameters  $H \in (0, 1)$  was treated in [29], where the authors prove strong uniqueness for locally unbounded drift coefficients in the case  $H < \frac{1}{2}$  by invoking a method based on the comparison theorem. See also [28].

Another crucial objective of our article is the study of the regularity of stochastic flows of the SDE (1.1), that is the regularity of

$$(x \mapsto X_t^x)$$

in the initial condition  $x \in \mathbb{R}^d$ , when the vector field  $b$  is discontinuous.

The motivation for this study comes from the deterministic case:

$$\frac{d}{dt}X_t^x = u(t, X_t^x), \quad t \geq 0, \quad X_0^x = x, \quad (1.2)$$

where  $u : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a vector field.

Here the solution  $X : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  to (1.2) may e.g. stand for the flow of fluid particles with respect to the velocity field of an incompressible inviscid fluid whose dynamics is described by an incompressible Euler equation

$$u_t + (Du)u + \nabla P = 0, \quad \nabla \cdot u = 0, \quad (1.3)$$

where  $P : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$  is the pressure field.

Solutions of (1.3) may be singular. Therefore a better understanding of the regularity of solutions of equation (1.3) requires the study of flows of ODE's (1.2) driven by irregular vector fields.

If  $u$  is Lipschitz continuous it is well-known that the unique flow  $X : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  in (1.2) is Lipschitzian. The latter classical result was generalized by Di Perna and Lions in their celebrated paper [8] to the case of time-homogeneous  $u \in W^{1,p}$  and  $\nabla \cdot u \in L^\infty$ , for which the authors construct a unique flow  $X$  to (1.2). Later on the latter result was extended by Ambrosio [2] to the case of vector fields of bounded variation.

However, it turns out that the superposition of the ODE (1.2) by a Brownian noise  $B$ , that is

$$dX_t = u(t, X_t)dt + dB_t, \quad s, t \geq 0, \quad X_s = x \in \mathbb{R}^d \quad (1.4)$$

has a strong regularising effect on its flow  $\mathbb{R}^d \ni x \mapsto \varphi_{s,t}(x) \in \mathbb{R}^d$ .

Using techniques similar to those in this paper, but without arguments based on local time, it was shown in Mohammed, Nilssen, Proske [26] for merely *bounded measurable* drift coefficients  $u$  that  $\varphi_{s,t}$  is a stochastic flow of Sobolev diffeomorphisms with

$$\varphi_{s,t}(\cdot), \varphi_{s,t}^{-1}(\cdot) \in L^2(\Omega, W^{1,p}(\mathbb{R}^d; w))$$

for all  $s, t$  and  $p \in (1, \infty)$ , where  $W^{1,p}(\mathbb{R}^d; w)$  is a weighted Sobolev space with weight function  $w : \mathbb{R}^d \rightarrow [0, \infty)$ .

As an application of this result the authors constructed Sobolev differentiable unique (weak) solutions of the (Stratonovich) stochastic transport equation with multiplicative noise of the form

$$\begin{cases} d_t v(t, x) + (u(t, x) \cdot Dv(t, x))dt + \sum_{i=1}^d e_i \cdot Dv(t, x) \circ dB_t^i = 0 \\ u(0, x) = u_0(x), \end{cases}$$

where  $u$  is bounded and measurable,  $u_0 \in C_b^1$  and where  $\{e_i\}_{i=1}^d$  is a basis of  $\mathbb{R}^d$ .

By adopting ideas in Mohammed et al. [26], we mention that the latter result on the existence of stochastic flows of Sobolev diffeomorphisms was extended in [33] to the case of globally integrable  $u \in L^{r,q}$  for  $r/d + 2/q < 1$  ( $r$  for the spatial variable and  $q$  for the temporal variable) and applied to the study of the regularity of solutions to Navier-Stokes equations. Compare also to [9], where the authors employ techniques based on solutions of backward Kolmogorov equations.

If the Brownian motion in (1.4) is replaced by a rougher noise given by  $B^H$  for  $H < \frac{1}{2}$ , we find in this paper for  $u \in L^\infty([0, T]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$  the rather surprising result which generalises the classical result of Kunita [18] for smooth coefficients, that the stochastic flow  $X : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is higher order Fréchet differentiable in the spatial variable, that is

$$(x \mapsto X_t^x(\omega)) \in C^k(\mathbb{R}^d)$$

a.s. for all  $t$  and for  $k \geq 1$ , provided  $H = H(k)$  is small enough.

In view of the above discussion in the case of Brownian noise driven stochastic flows, the latter result raises the fundamental question whether rough noise in the sense of  $B^H$  or a related noise with very irregular path behaviour may considerably regularise solutions of PDE's as e.g. transport equations, conservation laws or even Navier-Stokes equations by perturbation. We are confident that there is an affirmative answer for a class of interesting PDE's.

Finally, we comment on that the method for the construction of higher order Fréchet differentiable stochastic flows of (1.1), which is- as mentioned above- different from common techniques based on Markov processes and semimartingales, is inspired by the works [25], [23], [26], [14] in the case of (1.1) with initial Lévy noise and [10], [27] in the case of stochastic partial differential equations.

More precisely, in order to construct strong solutions to (1.1) we apply a compactness criterion for square integrable Brownian functionals in [5] to solutions  $X_t^n$  of

$$dX_t^n = b_n(t, X_t^n)dt + dB_t^H,$$

where  $b_n, n \geq 0$  are smooth coefficients converging to  $b$  in  $L^\infty([0, T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$  and show that  $X_t^n$  converges to a solution  $X_t$  of (1.1) in  $L^2(\Omega)$  for all  $t$ .

If, for a moment, we assume that  $b$  is time-homogeneous, then in proving the existence and the higher order Fréchet differentiability of the corresponding stochastic flow we make use of a "local time variational calculus" argument of the form

$$\int_{\Delta_{\theta,t}^m} \kappa(s) D^\alpha f(B_s^H) ds = \int_{\mathbb{R}^{dm}} D^\alpha f(z) L_\kappa(t, z) dz = (-1)^{|\alpha|} \int_{\mathbb{R}^{dm}} f(z) D^\alpha L_\kappa(t, z) dz, \quad (1.5)$$

for  $B_s^H = (B_{s_1}^H, \dots, B_{s_m}^H)$  and smooth functions  $f : \mathbb{R}^{dm} \rightarrow \mathbb{R}$ , where  $L_\kappa(t, z)$  is a spatially differentiable local time on the simplex  $\Delta_{\theta,t}^m = \{(s_1, \dots, s_m) \in [0, T]^m : \theta < s_1 < \dots < s_m < t\}$ , scaled by a function  $\kappa(s_1, \dots, s_m)$  ( $D^\alpha$  is the partial derivative of order  $|\alpha|$ ). Actually, we generalise the above argument to time dependent smooth functions  $f : [0, T] \times \mathbb{R}^{dm} \rightarrow \mathbb{R}$  and hence the intuition of the above "local time" argument is somehow not tangible any longer. In other words, we show that there exists a well-defined object  $\Lambda_\alpha^f(\theta, t, z)$  in  $L^2(\Omega)$  the size of which can be measured independently of the size of  $D^\alpha f$  such that the following integration by parts formula holds true

$$\int_{\Delta_{\theta,t}^m} D^\alpha f(s, B_s^H) ds = \int_{\mathbb{R}^{dm}} \Lambda_\alpha^f(\theta, t, z) dz, \quad P - a.s. \quad (1.6)$$

where the above formula coincides with (1.5) for time-homogeneous functions.

We expect that our approach can be also applied to the study of solutions of the following stochastic equations:

$$dX_t = (AX_t + b(X_t))dt + Qd\mathbf{W}_t,$$

for (mild) solutions  $X_t$ , where  $A$  is a densely defined linear operator (of parabolic type) on a separable Hilbert space  $H$ ,  $b : H \rightarrow H$  is a irregular function,  $Q$  a Hilbert-Schmidt operator and  $\mathbf{W}^H$  a (non-Hölder continuous) "cylindrical" Gaussian noise.

On the other hand, using our method we may also examine equations of the type

$$dX_t = dA_t + dB_t^H,$$

where  $A_t$  is a process of bounded variation which arises from limits of the form

$$\lim_{n \rightarrow \infty} \int_0^t b_n(X_s) ds$$

for coefficients  $b_n, n \geq 0$ . See [3] in the Brownian case.

Our paper is organised as follows: In Section 2 we introduce the mathematical framework of the article and define in Section 3 the random field  $\Lambda_\alpha^f$  of (1.6), which we show to be high-order differentiable in the spatial variable for small Hurst parameters. In Section 4 we establish the existence of a unique strong solution to the SDE (1.1) under integrability conditions on the drift coefficient  $b$ . Section 5 is devoted to the study of the regularity properties of stochastic flows of (1.1).

## 2. FRAMEWORK

In this section we recollect some specifics on fractional calculus, Malliavin calculus for fractional Brownian noise and occupation measures which will be extensively used throughout the article. The reader might consult [22], [21] or [7] for a general theory on Malliavin calculus for Brownian motion and [30, Chapter 5] for fractional Brownian motion. Whereas for occupation measures one may review [11] or [15]. We present the results in one dimension for simplicity inasmuch as we will treat the multidimensional case.

**2.1. Fractional calculus.** We establish here some basic definitions and properties on fractional calculus. A general theory on this subject may be found in [34] and [19].

Let  $a, b \in \mathbb{R}$  with  $a < b$ . Let  $f \in L^p([a, b])$  with  $p \geq 1$  and  $\alpha > 0$ . Define the *left-* and *right-sided Riemann-Liouville fractional integrals* by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy$$

and

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy$$

for almost all  $x \in [a, b]$  where  $\Gamma$  is the Gamma function.

Moreover, for a given integer  $p \geq 1$ , let  $I_{a+}^{\alpha}(L^p)$  (resp.  $I_{b-}^{\alpha}(L^p)$ ) denote the image of  $L^p([a, b])$  by the operator  $I_{a+}^{\alpha}$  (resp.  $I_{b-}^{\alpha}$ ). If  $f \in I_{a+}^{\alpha}(L^p)$  (resp.  $f \in I_{b-}^{\alpha}(L^p)$ ) and  $0 < \alpha < 1$  then define the *left-* and *right-sided Riemann-Liouville fractional derivatives* by

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^{\alpha}} dy$$

and

$$D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(y)}{(y-x)^{\alpha}} dy.$$

The left- and right-sided derivatives of  $f$  defined above have the following representations

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right)$$

and

$$D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} dy \right).$$

Finally, observe that by construction, the following formulas hold

$$I_{a+}^{\alpha}(D_{a+}^{\alpha} f) = f$$

for all  $f \in I_{a+}^{\alpha}(L^p)$  and

$$D_{a+}^{\alpha}(I_{a+}^{\alpha} f) = f$$

for all  $f \in L^p([a, b])$  and similarly for  $I_{b-}^{\alpha}$  and  $D_{b-}^{\alpha}$ .

**2.2. Shuffles.** Let  $m$  and  $n$  be integers. We define  $S(m, n)$  as the set of *shuffle permutations*, i.e. the set of permutations  $\sigma : \{1, \dots, m+n\} \rightarrow \{1, \dots, m+n\}$  such that  $\sigma(1) < \dots < \sigma(m)$  and  $\sigma(m+1) < \dots < \sigma(m+n)$ .

We define the  $m$ -dimensional simplex

$$\Delta_{\theta,t}^m := \{(s_m, \dots, s_1) \in [0, T]^m : \theta < s_m < \dots < s_1 < t\}.$$

The product of two simplices can be written as the following union

$$\Delta_{\theta,t}^m \times \Delta_{\theta,t}^n = \bigcup_{\sigma \in S(m,n)} \{(w_{m+n}, \dots, w_1) \in [0, T]^{m+n} : \theta < w_{\sigma(m+n)} < \dots < w_{\sigma(1)} < t\} \cup \mathcal{N},$$

where the set  $\mathcal{N}$  has null Lebesgue measure. In this way, if  $f_i : [0, T] \rightarrow \mathbb{R}$ ,  $i = 1, \dots, m+n$  are integrable functions we have

$$\begin{aligned} \int_{\Delta_{\theta,t}^m} \prod_{j=1}^m f_j(s_j) ds_m \dots ds_1 \int_{\Delta_{\theta,t}^n} \prod_{j=m+1}^{m+n} f_j(s_j) ds_{m+n} \dots ds_{m+1} \\ = \sum_{\sigma \in S(m,n)} \int_{\Delta_{\theta,t}^{m+n}} \prod_{j=1}^{m+n} f_{\sigma(j)}(w_j) dw_{m+n} \dots dw_1. \end{aligned} \quad (2.1)$$

We can generalize the above in technical lemma, the use of which shall be clear in Section 5. The reader is encouraged to skip this lemma and proof until Section 5.

**Lemma 2.1.** *Let  $n, p$  and  $k$  be integers,  $k \leq n$ . Assume we have integrable functions  $f_j : [0, T] \rightarrow \mathbb{R}$ ,  $j = 1, \dots, n$  and  $g_i : [0, T] \rightarrow \mathbb{R}$ ,  $i = 1, \dots, p$ . We may then write*

$$\begin{aligned} \int_{\Delta_{\theta,t}^n} f_1(s_1) \dots f_k(s_k) \int_{\Delta_{\theta,s_k}^p} g_1(r_1) \dots g_p(r_p) dr_p \dots dr_1 f_{k+1}(s_{k+1}) \dots f_n(s_n) ds_n \dots ds_1 \\ = \sum_{\sigma \in A_{n,p}} \int_{\Delta_{\theta,t}^{n+p}} h_1^\sigma(w_1) \dots h_{n+p}^\sigma(w_{n+p}) dw_{n+p} \dots dw_1, \end{aligned}$$

where  $h_i^\sigma \in \{f_j, g_i : 1 \leq j \leq n, 1 \leq i \leq p\}$ . Above  $A_{n,p}$  denotes a subset of permutations of  $\{1, \dots, n+p\}$  such that  $\#A_{n,p} \leq C^{n+p}$  for an appropriate constant  $C \geq 1$ , and we have defined  $s_0 = \theta$ .

*Proof.* The result is proved by induction on  $n$ . For  $n = 1$  and  $k = 0$  the result is trivial. For  $k = 1$  we have

$$\begin{aligned} \int_{\theta}^t f_1(s_1) \int_{\Delta_{\theta,s_1}^p} g_1(r_1) \dots g_p(r_p) dr_p \dots dr_1 ds_1 \\ = \int_{\Delta_{\theta,t}^{p+1}} f_1(w_1) g_1(w_2) \dots g_p(w_{p+1}) dw_{p+1} \dots dw_1, \end{aligned}$$

where we have put  $w_1 = s_1, w_2 = r_1, \dots, w_{p+1} = r_p$ .

Assume the result holds for  $n$  and let us show that this implies that the result is true for  $n+1$ . Either  $k = 0, 1$  or  $2 \leq k \leq n+1$ . For  $k = 0$  the result is trivial. For  $k = 1$  we

have

$$\begin{aligned} & \int_{\Delta_{\theta,t}^{n+1}} f_1(s_1) \int_{\Delta_{\theta,s_1}^p} g_1(r_1) \dots g_p(r_p) dr_p \dots dr_1 f_2(s_2) \dots f_{n+1}(s_{n+1}) ds_{n+1} \dots ds_1 \\ &= \int_{\theta}^t f_1(s_1) \left( \int_{\Delta_{\theta,s_1}^n} \int_{\Delta_{\theta,s_1}^p} g_1(r_1) \dots g_p(r_p) dr_p \dots dr_1 f_2(s_2) \dots f_{n+1}(s_{n+1}) ds_{n+1} \dots ds_2 \right) ds_1. \end{aligned}$$

The result follows from (2.1) coupled with  $\#S(n, p) = \frac{(n+p)!}{n!p!} \leq C^{n+p} \leq C^{(n+1)+p}$ . For  $k \geq 2$  we have from the induction hypothesis

$$\begin{aligned} & \int_{\Delta_{\theta,t}^{n+1}} f_1(s_1) \dots f_k(s_k) \int_{\Delta_{\theta,s_k}^p} g_1(r_1) \dots g_p(r_p) dr_p \dots dr_1 f_{k+1}(s_{k+1}) \dots f_{n+1}(s_{n+1}) ds_{n+1} \dots ds_1 \\ &= \int_{\theta}^t f_1(s_1) \int_{\Delta_{\theta,s_1}^n} f_2(s_2) \dots f_k(s_k) \int_{\Delta_{\theta,s_k}^p} g_1(r_1) \dots g_p(r_p) dr_p \dots dr_1 \\ & \quad \times f_{k+1}(s_{k+1}) \dots f_{n+1}(s_{n+1}) ds_{n+1} \dots ds_2 ds_1 \\ &= \sum_{\sigma \in A_{n,p}} \int_{\theta}^t f_1(s_1) \int_{\Delta_{\theta,s_1}^{n+p}} h_1^{\sigma}(w_1) \dots h_{n+p}^{\sigma}(w_{n+p}) dw_{n+p} \dots dw_1 ds_1 \\ &= \sum_{\tilde{\sigma} \in A_{n+1,p}} \int_{\Delta_{\theta,t}^{n+1+p}} h_1^{\tilde{\sigma}}(w_1) \dots h_{w_{n+1+p}}^{\tilde{\sigma}} dw_1 \dots dw_{n+1+p}, \end{aligned}$$

where  $A_{n+1,p}$  is the set of permutations  $\tilde{\sigma}$  of  $\{1, \dots, n+1+p\}$  such that  $\tilde{\sigma}(1) = 1$  and  $\tilde{\sigma}(j+1) = \sigma(j)$ ,  $j = 1, \dots, n+p$  for some  $\sigma \in A_{n,p}$ .  $\square$

*Remark 2.2.* Notice that the set  $A_{n,p}$  in the above lemma also depends on  $k$  but we shall not need this fact.

**2.3. Fractional Brownian motion.** Let  $B^H = \{B_t^H, t \in [0, T]\}$  be a  $d$ -dimensional *fractional Brownian motion* with Hurst parameter  $H \in (0, 1/2)$ . In other words,  $B^H$  is a centered Gaussian process with covariance structure

$$(R_H(t, s))_{i,j} := E[B_t^{H,(i)} B_s^{H,(j)}] = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}), \quad i, j = 1, \dots, d.$$

Observe that  $E[|B_t^H - B_s^H|^2] = d|t-s|^{2H}$  and hence  $B^H$  has stationary increments and Hölder continuous trajectories of index  $H - \varepsilon$  for all  $\varepsilon \in (0, H)$ . Observe moreover that the increments of  $B^H$ ,  $H \in (0, 1/2)$  are not independent. This fact makes computations more difficult. Another difficulty one encounters is that  $B^H$  is not a semimartingale, see e.g. [30, Proposition 5.1.1].

Now we give a brief survey on how to construct fractional Brownian motion via an isometry. Since the construction can be done componentwise we present here for simplicity the one-dimensional case. Further details can be found in [30].

Denote by  $\mathcal{E}$  the set of step functions on  $[0, T]$  and denote by  $\mathcal{H}$  the Hilbert space defined as the closure of  $\mathcal{E}$  with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$



The mapping  $1_{[0,t]} \mapsto B_t$  can be extended to an isometry between  $\mathcal{H}$  and the Gaussian subspace of  $L^2(\Omega)$  associated with  $B^H$ . Denote such isometry by  $\varphi \mapsto B^H(\varphi)$ . We recall the following result (see [30, Proposition 5.1.3]) which gives an integral representation of  $R_H(t, s)$  when  $H < 1/2$ :

**Proposition 2.3.** *Let  $H < 1/2$ . The kernel*

$$K_H(t, s) = c_H \left[ \left( \frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left( H - \frac{1}{2} \right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right],$$

where  $c_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H, H+1/2)}}$  being  $\beta$  the Beta function, satisfies

$$R_H(t, s) = \int_0^{t \wedge s} K_H(t, u) K_H(s, u) du. \quad (2.2)$$

The kernel  $K_H$  can also be represented by means of fractional derivatives as follows

$$K_H(t, s) = c_H \Gamma \left( H + \frac{1}{2} \right) s^{\frac{1}{2}-H} \left( D_{t-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} \right) (s).$$

Consider the linear operator  $K_H^* : \mathcal{E} \rightarrow L^2([0, T])$  defined by

$$(K_H^* \varphi)(s) = K_H(T, s) \varphi(s) + \int_s^T (\varphi(t) - \varphi(s)) \frac{\partial K_H}{\partial t}(t, s) dt$$

for every  $\varphi \in \mathcal{E}$ . Observe that  $(K_H^* 1_{[0,t]})(s) = K_H(t, s) 1_{[0,t]}(s)$ , then from this fact and (2.2) we see that  $K_H^*$  is an isometry between  $\mathcal{E}$  and  $L^2([0, T])$  which can be extended to the Hilbert space  $\mathcal{H}$ .

For a given  $\varphi \in \mathcal{H}$  one can show the following two representations for  $K_H^*$  in terms of fractional derivatives

$$(K_H^* \varphi)(s) = c_H \Gamma \left( H + \frac{1}{2} \right) s^{\frac{1}{2}-H} \left( D_{T-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} \varphi(u) \right) (s)$$

and

$$\begin{aligned} (K_H^* \varphi)(s) &= c_H \Gamma \left( H + \frac{1}{2} \right) \left( D_{T-}^{\frac{1}{2}-H} \varphi(s) \right) (s) \\ &\quad + c_H \left( \frac{1}{2} - H \right) \int_s^T \varphi(t) (t-s)^{H-\frac{3}{2}} \left( 1 - \left( \frac{t}{s} \right)^{H-\frac{1}{2}} \right) dt. \end{aligned}$$

One can show that  $\mathcal{H} = I_{T-}^{\frac{1}{2}-H}(L^2)$  (see [6] and [1, Proposition 6]).

Given the fact that  $K_H^*$  is an isometry from  $\mathcal{H}$  into  $L^2([0, T])$  the  $d$ -dimensional process  $W = \{W_t, t \in [0, T]\}$  defined by

$$W_t := B^H((K_H^*)^{-1}(1_{[0,t]})) \quad (2.3)$$

is a Wiener process and the process  $B^H$  has the following representation

$$B_t^H = \int_0^t K_H(t, s) dW_s, \quad (2.4)$$

see [1].

Henceforward, we will denote by  $W$  a standard Wiener process on a given probability space  $(\Omega, \mathfrak{A}, P)$  equipped with the natural filtration  $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$  generated by  $W$  augmented by all  $P$ -null sets and  $B := B^H$  the fractional Brownian motion with Hurst parameter  $H \in (0, 1/2)$  given by the representation (2.4).

Next, we give a version of Girsanov's theorem for fractional Brownian motion which is due to [6, Theorem 4.9]. Here we present the version given in [28, Theorem 3.1] but first we need to define an isomorphism  $K_H$  from  $L^2([0, T])$  onto  $I_{0+}^{H+\frac{1}{2}}(L^2)$  associated with the kernel  $K_H(t, s)$  in terms of the fractional integrals as follows, see [6, Theorem 2.1]

$$(K_H \varphi)(s) = I_{0+}^{2H} s^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} \varphi, \quad \varphi \in L^2([0, T]).$$

From this and the properties of the Riemann-Liouville fractional integrals and derivatives the inverse of  $K_H$  is given by

$$(K_H^{-1} \varphi)(s) = s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} D_{0+}^{2H} \varphi(s), \quad \varphi \in I_{0+}^{H+\frac{1}{2}}(L^2).$$

It follows that if  $\varphi$  is absolutely continuous, see [28], one can show that

$$(K_H^{-1} \varphi)(s) = s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} \varphi'(s). \quad (2.5)$$

**Theorem 2.4** (Girsanov's theorem for fBm). *Let  $u = \{u_t, t \in [0, T]\}$  be an  $\mathcal{F}$ -adapted process with integrable trajectories and set  $\tilde{B}_t^H = B_t^H + \int_0^t u_s ds$ ,  $t \in [0, T]$ . Assume that*

- (i)  $\int_0^\cdot u_s ds \in I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$ ,  $P$ -a.s.
- (ii)  $E[\xi_T] = 1$  where

$$\xi_T := \exp \left\{ - \int_0^T K_H^{-1} \left( \int_0^\cdot u_r dr \right) (s) dW_s - \frac{1}{2} \int_0^T K_H^{-1} \left( \int_0^\cdot u_r dr \right)^2 (s) ds \right\}.$$

*Then the shifted process  $\tilde{B}^H$  is an  $\mathcal{F}$ -fractional Brownian motion with Hurst parameter  $H$  under the new probability  $\tilde{P}$  defined by  $\frac{d\tilde{P}}{dP} = \xi_T$ .*

**Remark 2.5.** For the multidimensional case, define

$$(K_H \varphi)(s) := ((K_H \varphi^{(1)})(s), \dots, (K_H \varphi^{(d)})(s))^*, \quad \varphi \in L^2([0, T]; \mathbb{R}^d),$$

where  $*$  denotes transposition. Similarly for  $K_H^{-1}$  and  $K_H^*$ .

Finally, we will use a crucial property of the fractional Brownian motion which was proven by [31] for general Gaussian vector fields. This property will essentially help us to overcome the limitations of not having independent increments of the underlying noise.

Let  $m \in \mathbb{N}$  and  $0 =: t_0 < t_1 < \dots < t_m < T$ . Then for every  $\xi_1, \dots, \xi_m \in \mathbb{R}^d$  there exists a positive finite constant  $C > 0$  (not depending on  $m$ ) such that

$$\text{Var} \left[ \sum_{j=1}^m \langle \xi_j, B_{t_j} - B_{t_{j-1}} \rangle_{\mathbb{R}^d} \right] \geq C \sum_{j=1}^m |\xi_j|^2 \text{Var} [|B_{t_j} - B_{t_{j-1}}|^2]. \quad (2.6)$$

The above property is known as the (*strong*) *local non-determinism* property of the fractional Brownian motion. The reader may consult [31] or [36] for more information on this property.

## 3. AN INTEGRATION BY PARTS FORMULA

Let  $m$  be an integer and consider a  $f : [0, T]^m \times (\mathbb{R}^d)^m \rightarrow \mathbb{R}$  of the form

$$f(s, z) = \prod_{j=1}^m f_j(s_j, z_j), \quad s = (s_1, \dots, s_m) \in [0, T]^m, \quad z = (z_1, \dots, z_m) \in (\mathbb{R}^d)^m, \quad (3.1)$$

where  $f_j : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $j = 1, \dots, m$  are smooth functions with compact support. Moreover, consider an integrable  $\kappa : [0, T]^m \rightarrow \mathbb{R}$  of the form

$$\kappa(s) = \prod_{j=1}^m \kappa_j(s_j), \quad s \in [0, T]^m, \quad (3.2)$$

where  $\kappa_j : [0, T] \rightarrow \mathbb{R}$ ,  $j = 1, \dots, m$  are integrable functions.

Denote by  $\alpha_j$  a multiindex and  $D^{\alpha_j}$  its corresponding differential operator. For  $\alpha = (\alpha_1, \dots, \alpha_m)$  considered an element of  $\mathbb{N}_0^{d \times m}$  so that  $|\alpha| := \sum_{j=1}^m \sum_{l=1}^d \alpha_j^{(l)}$ , we write

$$D^\alpha f(s, z) = \prod_{j=1}^m D^{\alpha_j} f_j(s_j, z_j).$$

The aim of this section is to derive an integration by parts formula of the form

$$\int_{\Delta_{\theta, t}^m} D^\alpha f(s, B_s) ds = \int_{(\mathbb{R}^d)^m} \Lambda_\alpha^f(\theta, t, z) dz, \quad (3.3)$$

for a suitable random field  $\Lambda_\alpha^f$ . In fact, we have

$$\Lambda_\alpha^f(\theta, t, z) = (2\pi)^{-dm} \int_{(\mathbb{R}^d)^m} \int_{\Delta_{\theta, t}^m} \prod_{j=1}^m f_j(s_j, z_j) (-iu_j)^{\alpha_j} \exp\{-i\langle u_j, B_{s_j} - z_j \rangle\} ds du. \quad (3.4)$$

We start by *defining*  $\Lambda_\alpha^f(\theta, t, z)$  as above and show that it is a well-defined element of  $L^2(\Omega)$ .

Introduce the following notation: given  $(s, z) = (s_1, \dots, s_m, z_1, \dots, z_m) \in [0, T]^m \times (\mathbb{R}^d)^m$  and a shuffle  $\sigma \in S(m, m)$  we write

$$f_\sigma(s, z) := \prod_{j=1}^{2m} f_{[\sigma(j)]}(s_j, z_{[\sigma(j)]})$$

and

$$\kappa_\sigma(s) := \prod_{j=1}^{2m} \kappa_{[\sigma(j)]}(s_j)$$

where  $[\cdot] : \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$  is the usual congruence class function.

For an integer  $k \geq 0$  we define

$$\Psi_k^f(\theta, t, z) := \sum_{\sigma \in S(m, m)} \int_{\Delta_{\theta, t}^{2m}} |f_\sigma(s, z)| \prod_{j=1}^{2m} |s_j - s_{j-1}|^{-dH(2k+1)} ds,$$

respectively,

$$\Psi_k^\kappa(\theta, t) := \sum_{\sigma \in S(m, m)} \int_{\Delta_{\theta, t}^{2m}} |\kappa_\sigma(s)| \prod_{j=1}^{2m} |s_j - s_{j-1}|^{-dH(2k+1)} ds.$$

**Theorem 3.1.** *Suppose  $\alpha$  is such that  $\alpha_j^{(l)} \leq k$  for all  $j = 1, \dots, m$  and  $l = 1, \dots, d$  and that  $\Psi_k^f(\theta, t, z) < \infty$ . Then, defining  $\Lambda_\alpha^f(\theta, t, z)$  as in (3.4) gives a random variable in  $L^2(\Omega)$  and there exists a universal constant  $C > 0$  such that*

$$E[|\Lambda_\alpha^f(\theta, t, z)|^2] \leq C^m \Psi_k^f(\theta, t, z). \quad (3.5)$$

Moreover, we have

$$\left| E \left[ \int_{(\mathbb{R}^d)^m} \Lambda_\alpha^{\kappa f}(\theta, t, z) dz \right] \right| \leq C^m \prod_{j=1}^m \|f_j\|_{L^\infty([0, T], L^1(\mathbb{R}^d))} \Psi_k^\kappa(\theta, t)^{1/2}. \quad (3.6)$$

*Proof.* For notational convenience we consider  $\theta = 0$  and let  $\Lambda_\alpha^f(t, z) := \Lambda_\alpha^f(0, t, z)$ .

For an integrable function  $g : (\mathbb{R}^d)^m \rightarrow \mathbb{C}$  we can write

$$\begin{aligned} & \left| \int_{\mathbb{R}^{dm}} g(u_1, \dots, u_m) du_1 \dots du_m \right|^2 \\ &= \int_{\mathbb{R}^{dm}} g(u_1, \dots, u_m) du_1 \dots du_m \overline{\int_{\mathbb{R}^{dm}} g(u_{m+1}, \dots, u_{2m}) du_{m+1} \dots du_{2m}} \\ &= \int_{\mathbb{R}^{dm}} g(u_1, \dots, u_m) du_1 \dots du_m \int_{\mathbb{R}^{dm}} \overline{g(u_{m+1}, \dots, u_{2m})} du_{m+1} \dots du_{2m} \\ &= \int_{\mathbb{R}^{dm}} g(u_1, \dots, u_m) du_1 \dots du_m (-1)^{dm} \int_{\mathbb{R}^{dm}} \overline{g(-u_{m+1}, \dots, -u_{2m})} du_{m+1} \dots du_{2m} \\ &= (-1)^{dm} \int_{\mathbb{R}^{2dm}} g(u_1, \dots, u_m) \overline{g(-u_{m+1}, \dots, -u_{2m})} du_1 \dots du_{2m}, \end{aligned}$$

where we have used the change of variables  $(u_{m+1}, \dots, u_{2m}) \mapsto (-u_{m+1}, \dots, -u_{2m})$  in the second to last equality.

This gives

$$\begin{aligned}
& |\Lambda_\alpha^f(t, z)|^2 \\
&= (2\pi)^{-2dm} (-1)^{dm} \int_{(\mathbb{R}^d)^{2m}} \int_{\Delta_{0,t}^m} \prod_{j=1}^m f_j(s_j, z_j) (-iu_j)^{\alpha_j} e^{-i\langle u_j, B_{s_j} - z_j \rangle} ds_1 \cdots ds_m \\
&\quad \times \int_{\Delta_{0,t}^m} \prod_{j=m+1}^{2m} f_{[j]}(s_j, z_{[j]}) (-iu_j)^{\alpha_j} e^{-i\langle u_j, B_{s_j} - z_{[j]} \rangle} ds_{m+1} \cdots ds_{2m} du_1 \cdots du_{2m} \\
&= (2\pi)^{-2dm} \int_{(\mathbb{R}^d)^{2m}} \left( \prod_{j=1}^m e^{i\langle z_j, u_j + u_{j+m} \rangle} u_j^{\alpha_j} u_{j+m}^{\alpha_j} \right) \\
&\quad \times \int_{\Delta_{0,t}^m} \prod_{j=1}^m f_j(s_j, z_j) e^{-i\langle u_j, B_{s_j} \rangle} ds_1 \cdots ds_m \\
&\quad \times \int_{\Delta_{0,t}^m} \prod_{j=m+1}^{2m} f_{[j]}(s_j, z_{[j]}) e^{-i\langle u_j, B_{s_j} \rangle} ds_{m+1} \cdots ds_{2m} du_1 \cdots du_{2m} \\
&= (2\pi)^{-2dm} \sum_{\sigma \in S(m, m)} \int_{(\mathbb{R}^d)^{2m}} \left( \prod_{j=1}^m e^{i\langle z_j, u_j + u_{j+m} \rangle} u_j^{\alpha_j} u_{j+m}^{\alpha_j} \right) \\
&\quad \times \int_{\Delta_{0,t}^{2m}} f_\sigma(s, z) \exp \left\{ -i \sum_{j=1}^{2m} \langle u_{\sigma(j)}, B_{s_j} \rangle \right\} ds_1 \cdots ds_{2m} du_1 \cdots du_{2m},
\end{aligned}$$

where we have used (2.1) in the last step.

Fix  $\sigma \in S(m, m)$  and consider each of the terms in the above sum. Denote by  $P_\sigma$  the linear transformation on  $(\mathbb{R}^d)^{2m}$  such that  $u = P_\sigma u_\sigma$ , where  $u_\sigma := (u_{\sigma(1)}, \dots, u_{\sigma(2m)})$  and  $M$  such that  $v = M\zeta$  where  $v_j = \zeta_j - \zeta_{j+1}$ ,  $j = 1, \dots, 2m-1$ ,  $v_{2m} = \zeta_{2m}$ .

We apply the change of variables  $u = P_\sigma^{-1} M\zeta$  and set  $s_0 := 0$  in order to get

$$\begin{aligned}
& \int_{(\mathbb{R}^d)^{2m}} \prod_{j=1}^m e^{i\langle z_j, u_j + u_{j+m} \rangle} u_j^{\alpha_j} u_{j+m}^{\alpha_j} \int_{\Delta_{0,t}^{2m}} f_\sigma(s, z) \exp \left\{ -i \sum_{j=1}^{2m} \langle u_{\sigma(j)}, B_{s_j} \rangle \right\} ds_1 \cdots ds_{2m} du_1 \cdots du_{2m} \\
&= \int_{(\mathbb{R}^d)^{2m}} \left( \prod_{j=1}^m e^{i\langle z_j, (P_\sigma^{-1} M\zeta)_j + (P_\sigma^{-1} M\zeta)_{j+m} \rangle} (P_\sigma^{-1} M\zeta)_j^{\alpha_j} (P_\sigma^{-1} M\zeta)_{j+m}^{\alpha_j} \right) \\
&\quad \times \int_{\Delta_{0,t}^{2m}} f_\sigma(s, z) \exp \left\{ -i \sum_{j=1}^{2m} \langle \zeta_j, B_{s_j} - B_{s_{j-1}} \rangle \right\} ds_1 \cdots ds_{2m} d\zeta_1 \cdots d\zeta_{2m}.
\end{aligned}$$

The expected value of the above is bounded by

$$\begin{aligned}
& \int_{(\mathbb{R}^d)^{2m}} \prod_{j=1}^m |(P_\sigma^{-1} M\zeta)_j^{\alpha_j} (P_\sigma^{-1} M\zeta)_{j+m}^{\alpha_j}| \\
& \quad \times \int_{\Delta_{0,t}^{2m}} |f_\sigma(s, z)| E[\exp\{-i \sum_{j=1}^{2m} \langle \zeta_j, B_{s_j} - B_{s_{j-1}} \rangle\}] ds_1 \cdots ds_{2m} d\zeta_1 \cdots d\zeta_{2m} \\
& = \int_{(\mathbb{R}^d)^{2m}} \prod_{j=1}^m |(P_\sigma^{-1} M\zeta)_j^{\alpha_j} (P_\sigma^{-1} M\zeta)_{j+m}^{\alpha_j}| \\
& \quad \times \int_{\Delta_{0,t}^{2m}} |f_\sigma(s, z)| \exp\left\{-\frac{1}{2} \text{Var}\left(\sum_{j=1}^{2m} \langle \zeta_j, B_{s_j} - B_{s_{j-1}} \rangle\right)\right\} ds_1 \cdots ds_{2m} d\zeta_1 \cdots d\zeta_{2m} \\
& \leq \int_{(\mathbb{R}^d)^{2m}} \prod_{j=1}^m |(P_\sigma^{-1} M\zeta)_j^{\alpha_j} (P_\sigma^{-1} M\zeta)_{j+m}^{\alpha_j}| \\
& \quad \times \int_{\Delta_{0,t}^{2m}} |f_\sigma(s, z)| \exp\left\{-\frac{C}{2} \sum_{j=1}^{2m} |\zeta_j|^2 |s_j - s_{j-1}|^{2H}\right\} ds_1 \cdots ds_{2m} d\zeta_1 \cdots d\zeta_{2m},
\end{aligned}$$

where we have used (2.6).

Write

$$\prod_{j=1}^m |(P_\sigma^{-1} M\zeta)_j^{\alpha_j} (P_\sigma^{-1} M\zeta)_{j+m}^{\alpha_j}| = \prod_{j=1}^{2m} |(\zeta_j - \zeta_{j+1})^{\tilde{\alpha}_j}|,$$

where  $\tilde{\alpha}_j := \alpha_{[\sigma^{-1}(j)]}$ .

Now observe that we can express the above product as a sum of different combinations where the exponent is, at most, two. That is

$$\prod_{j=1}^{2m} \prod_{l=1}^d |\zeta_j^{(l)} - \zeta_{j+1}^{(l)}|^{\tilde{\alpha}_j^{(l)}} = \sum_{\substack{\delta \in I \\ \delta \in \{0,1,2\}^{2m}}} c_\delta \prod_{j=1}^{2m} \prod_{l=1}^d |\zeta_j^{(l)}|^{\delta_j \tilde{\alpha}_j^{(l)}}$$

for some constants  $c_\delta$  and here  $I$  is a set of indices which has  $2^m$  elements. Now, we have that the integral w.r.t.  $\zeta$  can be written as

$$\begin{aligned}
A &:= \int_{(\mathbb{R}^d)^{2m}} \prod_{j=1}^{2m} \prod_{l=1}^d |\zeta_j^{(l)} - \zeta_{j+1}^{(l)}|^{\tilde{\alpha}_j^{(l)}} e^{-\frac{C}{2} \sum_{j=0}^{2m} |\zeta_j|^2 |s_j - s_{j-1}|^{2H}} d\zeta \\
&= \sum_{\substack{\delta \in I \\ \delta \in \{0,1,2\}^{2m}}} c_\delta \prod_{j=1}^{2m} \prod_{l=1}^d \left( \int_{\mathbb{R}} |\zeta_j^{(l)}|^{\delta_j \tilde{\alpha}_j^{(l)}} e^{-\frac{1}{2\sigma_j^2} |\zeta_j^{(l)}|^2} d\zeta_j^{(l)} \right),
\end{aligned}$$

where for each  $j = 1, \dots, 2m$ ,

$$\sigma_j^2 := \frac{1}{C} |s_j - s_{j-1}|^{-2H}.$$

Then

$$\begin{aligned}
 A &= \sum_{\substack{\delta \in I \\ \delta \in \{0,1,2\}^{2m}}} c_\delta \prod_{j=1}^{2m} \prod_{l=1}^d \left( \frac{\sqrt{2\pi\sigma_j^2} \sigma_j^{\delta_j \tilde{\alpha}_j^{(l)}} 2^{\delta_j \tilde{\alpha}_j^{(l)}/2} \Gamma\left(\frac{\delta_j \tilde{\alpha}_j^{(l)} + 1}{2}\right)}{\sqrt{\pi}} \right) \\
 &\leq C^m \sum_{\substack{\delta \in I \\ \delta \in \{0,1,2\}^{2m}}} \prod_{j=1}^{2m} \sigma_j^{\delta_j \sum_{l=1}^d \tilde{\alpha}_j^{(l)} + d} \\
 &= C^m \sum_{\substack{\delta \in I \\ \delta \in \{0,1,2\}^{2m}}} \prod_{j=1}^{2m} \prod_{l=1}^d |s_j - s_{j-1}|^{-H(\delta_j \sum_{l=1}^d \tilde{\alpha}_j^{(l)} + d)} \\
 &\leq C^m \prod_{j=1}^{2m} |s_j - s_{j-1}|^{-dH(2k+1)} 1_{\{|s_j - s_{j-1}| < 1\}} \\
 &\quad + C^m 1_{\{|s_j - s_{j-1}| > 1\}}
 \end{aligned}$$

for some constant  $C > 0$  depending only on  $d$  and  $k$  where we used  $\sum_{l=1}^d \tilde{\alpha}_j^{(l)} \leq kd$  and  $\delta_j \leq 2$  for every  $j = 1, \dots, 2m$ . The second term is clearly integrable w.r.t.  $s$ .

The result follows.

Finally, we show estimate (3.6). Taking modulus inside the integral and using estimate (3.5) we have

$$\left| E \left[ \int_{(\mathbb{R}^d)^m} \Lambda_\alpha^{\kappa f}(\theta, t, z) dz \right] \right| \leq C^m \int_{(\mathbb{R}^d)^m} \Psi_k^{\kappa f}(\theta, t, z)^{1/2} dz.$$

Taking the supremum over  $[0, T]$  for each function  $f_j$ , i.e.

$$|f_{[\sigma(j)]}(s_j, z_{[\sigma(j)]})| \leq \sup_{s_j \in [0, T]} |f_{[\sigma(j)]}(s_j, z_{[\sigma(j)]})|, \quad j = 1, \dots, 2m,$$

one obtains

$$\begin{aligned}
 &\left| E \left[ \int_{(\mathbb{R}^d)^m} \Lambda_\alpha^{\kappa f}(\theta, t, z) dz \right] \right| \\
 &\leq C^m \prod_{j=1}^m \|f_j\|_{L^\infty([0, T], L^1(\mathbb{R}^d))} \left( \sum_{\sigma \in S(m, m)} \int_{\Delta^{2m}(\theta, t)} |\kappa_\sigma(s)| \prod_{j=1}^{2m} |s_j - s_{j-1}|^{-dH(2k+1)} ds \right)^{1/2}.
 \end{aligned}$$

□

We remark that *a priori* one can not interchange the order of integration in (3.4). Indeed, for  $m = 1$ ,  $f \equiv 1$  one gets an integral of the Donsker-Delta function which is not a random variable in the usual sense. To overcome this define for  $R > 0$ ,

$$\Lambda_{\alpha, R}^f(\theta, t, z) := (2\pi)^{-dm} \int_{B(0, R)} \int_{\Delta_{\theta, t}^m} \prod_{j=1}^m f_j(s_j, z_j) (-iu_j)^{\alpha_j} e^{-i\langle u_j, B_{s_j} - z_j \rangle} ds dv,$$

where  $B(0, R) := \{v \in \mathbb{R}^{dm} : |v| < R\}$ . Clearly we have

$$|\Lambda_{\alpha, R}^f(\theta, t, z)| \leq C_R \int_{\Delta_{\theta, t}^m} \prod_{j=1}^m |f_j(s_j, z_j)| ds$$

for an appropriate constant  $C_R$ . Let us assume that the above right-hand side is integrable over  $(\mathbb{R}^d)^m$ .

Similar computations as above show that  $\Lambda_{\alpha, R}^f(\theta, t, z) \rightarrow \Lambda_{\alpha}^f(\theta, t, z)$  in  $L^2(\Omega)$  as  $R \rightarrow \infty$  for all  $\theta, t$  and  $z$ .

Lebesgue's dominated convergence theorem and the fact that the Fourier transform is an automorphism on the Schwarz space yield

$$\begin{aligned} \int_{\mathbb{R}^{dm}} \Lambda_{\alpha}^f(\theta, t, z) dz &= \lim_{R \rightarrow \infty} \int_{\mathbb{R}^{dm}} \Lambda_{\alpha, R}^f(\theta, t, z) dz \\ &= \lim_{R \rightarrow \infty} (2\pi)^{-dm} \int_{\mathbb{R}^{dm}} \int_{B(0, R)} \int_{\Delta_{\theta, t}^m} \prod_{j=1}^m f_j(s_j, z_j) (-iu_j)^{\alpha_j} e^{-i\langle u_j, B_{s_j} - z_j \rangle_{\mathbb{R}^d}} dz du ds \\ &= \lim_{R \rightarrow \infty} \int_{\Delta_{\theta, t}^m} \int_{B(0, R)} (2\pi)^{-dm} \int_{\mathbb{R}^{dm}} \prod_{j=1}^m f_j(s_j, z_j) (-iu_j)^{\alpha_j} e^{-i\langle u_j, B_{s_j} \rangle_{\mathbb{R}^d}} dz du ds \\ &= \lim_{R \rightarrow \infty} \int_{\Delta_{\theta, t}^m} \int_{B(0, R)} \prod_{j=1}^m \widehat{f}_j(s, -u_j) (-iu_j)^{\alpha_j} e^{-i\langle u_j, B_{s_j} \rangle_{\mathbb{R}^d}} du ds \\ &= \int_{\Delta_{\theta, t}^m} D^{\alpha} f(s, B_s) ds \end{aligned}$$

which is exactly (3.3).

Next, we give a crucial estimate which shows why fractional Brownian motion actually regularises (1.1). It is based on integration by parts and the aforementioned properties of the local-time  $L$ . The estimate we obtain can be presented in a more explicit way when

$$\kappa_j(s) = (K_H(s, \theta) - K_H(s, \theta'))^{\varepsilon_j}, \quad s \in [0, T]$$

for every  $j = 1, \dots, m$  or,

$$\kappa_j(s) = (K_H(s, \theta))^{\varepsilon_j}, \quad s \in [0, T]$$

for every  $j = 1, \dots, m$  with  $(\varepsilon_1, \dots, \varepsilon_m) \in \{0, 1\}^m$  and we will see why these are important in the next coming section.

The proof can be found in the Appendix, Lemma A.5 and Lemma A.6.

**Proposition 3.2.** *Let  $B^H$ ,  $H \in (0, 1/2)$ , be a standard  $d$ -dimensional fractional Brownian motion and functions  $f$  and  $\kappa$  as in (3.1), respectively as in (3.2). Let*

$$\kappa_j(s) = (K_H(s, \theta) - K_H(s, \theta'))^{\varepsilon_j}, \quad s \in [0, T]$$

*for every  $j = 1, \dots, m$  with  $(\varepsilon_1, \dots, \varepsilon_m) \in \{0, 1\}^m$  for  $\theta, \theta' \in [0, T]$  with  $\theta' < \theta$ . Let  $\alpha \in (\mathbb{N}_0^d)^m$  be an multi-index such that  $\alpha_i^{(j)} \leq k$  for all  $i = 1, \dots, m$  and  $j = 1, \dots, d$ . If*

$$H < \min_{m \geq 1} \frac{m - \frac{1}{2} \sum_{j=1}^m \varepsilon_j}{md(2k+1) - \sum_{j=1}^m \varepsilon_j}, \quad (3.7)$$



then, there exists a universal constant  $C > 0$  (independent of  $m$ ,  $\{f_i\}_{i=1,\dots,m}$  and  $\alpha$ ) such that for any  $\theta, t \in [0, T]$  with  $\theta < t$  we have

$$\begin{aligned} & \left| E \left[ \int_{\Delta_{\theta,t}^m} \left( \prod_{j=1}^m D^{\alpha_j} f_j(s_j, B_{s_j}^H) \kappa_j(s_j) \right) ds \right] \right| \\ & \leq C^m \prod_{j=1}^m \|f_j\|_{L^\infty([0,T], L^1(\mathbb{R}^d))} \frac{|\theta' - \theta|^{\gamma \sum_{j=1}^m \varepsilon_j} |t - \theta|^{m(1-d(2k+1)H) + (H - \frac{1}{2} - \gamma) \sum_{j=1}^m \varepsilon_j}}{\Gamma \left( 2m(1 - dH(2k+1)) + 1 + 2(H - \frac{1}{2} - \gamma) \sum_{j=1}^m \varepsilon_j \right)^{1/2}} \end{aligned} \quad (3.8)$$

for  $\gamma \in (0, H)$ .

*Proof.* By definition of  $\Lambda_\alpha^{\kappa f}$  in (3.4) it immediately follows that the integral in (3.8) can be expressed as

$$\int_{\Delta_{\theta,t}^m} \left( \prod_{j=1}^m D^{\alpha_j} f_j(s_j, B_{s_j}^H) \kappa_j(s_j) \right) ds = \int_{\mathbb{R}^{dm}} \Lambda_\alpha^{\kappa f}(\theta, t, z) dz.$$

Taking expectation and using estimate (3.6) we obtain

$$\left| E \int_{\Delta_{\theta,t}^m} \left( \prod_{j=1}^m D^{\alpha_j} f_j(s_j, B_{s_j}^H) \kappa_j(s_j) \right) ds \right| \leq C^m \prod_{j=1}^m \|f_j\|_{L^\infty([0,T], L^1(\mathbb{R}^d))} \Psi_k^\kappa(\theta, t)^{1/2}.$$

Finally, since  $H$  satisfies (3.7) Lemma A.5 in the Appendix allows us to conclude.  $\square$

**Proposition 3.3.** Let  $B^H$ ,  $H \in (0, 1/2)$ , be a standard  $d$ -dimensional fractional Brownian motion and functions  $f$  and  $\kappa$  as in (3.1), respectively as in (3.2). Let

$$\kappa_j(s) = (K_H(s, \theta))^{\varepsilon_j}, \quad s \in [0, T]$$

for every  $j = 1, \dots, m$  with  $(\varepsilon_1, \dots, \varepsilon_m) \in \{0, 1\}^m$  for  $\theta, \theta' \in [0, T]$  with  $\theta' < \theta$ . Let  $\alpha \in (\mathbb{N}_0^d)^m$  be an multi-index such that  $\alpha_i^{(j)} \leq k$  for all  $i = 1, \dots, m$  and  $j = 1, \dots, d$ . If

$$H < \min_{m \geq 1} \frac{m - \frac{1}{2} \sum_{j=1}^m \varepsilon_j}{md(2k+1) - \sum_{j=1}^m \varepsilon_j}, \quad (3.9)$$

then, there exists a universal constant  $C > 0$  (independent of  $m$ ,  $\{f_i\}_{i=1,\dots,m}$  and  $\alpha$ ) such that for any  $\theta, t \in [0, T]$  with  $\theta < t$  we have

$$\begin{aligned} & \left| E \left[ \int_{\Delta_{\theta,t}^m} \left( \prod_{j=1}^m D^{\alpha_j} f_j(s_j, B_{s_j}^H) \kappa_j(s_j) \right) ds \right] \right| \\ & \leq C^m \prod_{j=1}^m \|f_j\|_{L^\infty([0,T], L^1(\mathbb{R}^d))} \frac{|t - \theta|^{m(1-dH(2k+1)) + (H - \frac{1}{2}) \sum_{j=1}^m \varepsilon_j}}{\Gamma \left( 2m(1 - dH(2k+1)) + 1 + 2(H - \frac{1}{2}) \sum_{j=1}^m \varepsilon_j \right)^{1/2}}. \end{aligned} \quad (3.10)$$

*Proof.* Similar to the proof of Proposition 3.2 in connection with Lemma A.6 in the Appendix.  $\square$

## 4. EXISTENCE AND UNIQUENESS OF GLOBAL STRONG SOLUTIONS

As outlined in the introduction the object of study is a time-inhomogeneous SDE with additive  $d$ -dimensional fractional Brownian noise  $B^H$  with Hurst parameter  $H \in (0, 1/2)$ , i.e.

$$dX_t = b(t, X_t)dt + dB_t^H, \quad X_0 = x \in \mathbb{R}^d, \quad t \in [0, T], \quad (4.1)$$

where  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a Borel-measurable function. We will study equation (1.1) when the drift coefficient  $b$  belongs to  $L^\infty([0, T], L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ . We will introduce the following short notation for the following functional spaces

$$\begin{aligned} L_\infty^{1,\infty} &:= L^\infty([0, T]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) \\ L_\infty^1 &:= L^\infty([0, T]; L^1(\mathbb{R}^d)) \\ L_\infty^\infty &:= L^\infty([0, T]; L^\infty(\mathbb{R}^d)). \end{aligned}$$

Hence the subscript refers to the *supremum*'s norm on  $[0, T]$  whereas the superscript indicates the norm used for the space variable.

Hereunder, we establish the main result of this section.

**Theorem 4.1.** *Let  $b \in L_\infty^{1,\infty}$ . Then if  $H < \frac{1}{2(3d-1)}$ ,  $d \geq 1$  there exists a unique (global) strong solution  $X = \{X_t, t \in [0, T]\}$  of equation (1.1). Moreover, for every  $t \in [0, T]$ ,  $X_t$  is Malliavin differentiable in the direction of the Brownian motion  $W$  in (2.3).*

The proof of Theorem 4.1 is based on the following steps:

- (1) First, we construct a weak solution  $X$  to (1.1) by means of Girsanov's theorem, that is we introduce a probability space  $(\Omega, \mathfrak{A}, P)$  that carries a fractional Brownian motion  $B^H$  and a process  $X$  such that (1.1) is fulfilled. However, a priori  $X$  is not adapted to the filtration  $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$  generated by  $B^H$ .
- (2) Next, we approximate the drift coefficient  $b$  by a sequence of functions (which always exists by standard approximation results)  $b_n$ ,  $n \geq 0$  such that  $\{b_n\}_{n \geq 0} \subset C_c^\infty([0, T] \times \mathbb{R}^d)$  with  $\|b_n - b\|_{L_\infty^1} \rightarrow 0$  as  $n \rightarrow \infty$  and such that  $|b_n(t, x)| \leq M < \infty$ ,  $n \geq 0$  a.e. for some constant  $M$ . By standard results on SDEs, we know that for each smooth coefficient  $b_n$ ,  $n \geq 0$ , there exists unique strong solution  $X^n$  to the SDE

$$dX_t^n = b_n(t, X_t^n)dt + dB_t^H, \quad 0 \leq t \leq T, \quad X_0^n = x \in \mathbb{R}^d. \quad (4.2)$$

We then show that for each  $t \in [0, T]$  the sequence  $X_t^n$  converges weakly to the conditional expectation  $E[X_t | \mathcal{F}_t]$  in the space  $L^2(\Omega; \mathcal{F}_t)$  of square integrable,  $\mathcal{F}_t$ -measurable random variables.

- (3) It is well known, see e.g. [30], that for each  $t \in [0, T]$  the strong solution  $X_t^n$ ,  $n \geq 0$ , is Malliavin differentiable, and that the Malliavin derivative  $D_s X_t^n$ ,  $0 \leq s \leq t$ , with respect to  $W$  in (2.3) satisfies

$$D_s X_t^n = K_H(t, s)I_d + \int_s^t b'_n(u, X_u^n) D_s X_u^n du, \quad (4.3)$$

where  $b'_n$  denotes the Jacobian of  $b_n$ . In the next step we then employ a compactness criterion based on Malliavin calculus to show that for every  $t \in [0, T]$  the set

of random variables  $\{X_t^n\}_{n \geq 0}$  is relatively compact in  $L^2(\Omega)$ , which then admits the conclusion that  $X_t^n$  converges strongly in  $L^2(\Omega; \mathcal{F}_t)$  to  $E[X_t | \mathcal{F}_t]$ . Further we see that  $E[X_t | \mathcal{F}_t]$  is Malliavin differentiable as a consequence of the compactness criterion.

- (4) In the last step we show that  $E[X_t | \mathcal{F}_t] = X_t$ , which implies that  $X_t$  is  $\mathcal{F}_t$ -measurable and thus a strong solution on our specific probability space.

We turn to the first step of our scheme which is to construct weak solutions of (1.1) by using Girsanov's theorem in this context. Let  $(\Omega, \mathfrak{A}, \tilde{P})$  be some given probability space which carries a  $d$ -dimensional fractional Brownian motion  $\tilde{B}^H$  with Hurst parameter  $H \in (0, 1/2)$  and set  $X_t := x + \tilde{B}_t^H$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ . Set  $\theta_t := (K_H^{-1}(\int_0^t b(r, X_r) dr)) (t)$  and consider the Doléans-Dade exponential

$$Z_t := \mathcal{E}(\theta)_t := \exp \left\{ \int_0^t \theta_s^T dW_s - \frac{1}{2} \int_0^t \theta_s^T \theta_s ds \right\}, \quad t \in [0, T].$$

The following two lemmata show that the conditions of Theorem 2.4 hold.

**Lemma 4.2.** *Let  $\tilde{B}_t^H$  be a  $d$ -dimensional fractional Brownian motion with respect to  $(\Omega, \mathfrak{A}, \tilde{P})$ . Then*

$$\int_0^\cdot |b(s, \tilde{B}_s^H)| ds \in I_{0+}^{H+\frac{1}{2}}(L^2), \quad P - a.s.$$

*Proof.* Using the property that  $D_{0+}^{H+\frac{1}{2}} I_{0+}^{H+\frac{1}{2}}(f) = f$  for  $f \in L^2([0, T])$  we need to show that

$$D_{0+}^{H+\frac{1}{2}} \int_0^\cdot |b(s, \tilde{B}_s^H)| ds \in L^2([0, T]), \quad P - a.s.$$

Indeed,

$$\begin{aligned} \left| D_{0+}^{H+\frac{1}{2}} \left( \int_0^\cdot |b(s, \tilde{B}_s^H)| ds \right) (t) \right| &= \frac{1}{\Gamma(\frac{1}{2} - H)} \left( \frac{1}{t^{H+\frac{1}{2}}} \int_0^t |b(u, \tilde{B}_u^H)| du \right. \\ &\quad \left. + \left( H + \frac{1}{2} \right) \int_0^t (t-s)^{-H-\frac{3}{2}} \int_s^t |b(u, \tilde{B}_u^H)| ds \right) \\ &= \frac{t^{\frac{1}{2}-H}}{\Gamma(\frac{1}{2} - H)} \frac{1}{\frac{1}{2} - H} \|b\|_{L^\infty}. \end{aligned}$$

Hence, for some finite constant  $C_H > 0$  we have

$$\left| D_{0+}^{H+\frac{1}{2}} \left( \int_0^\cdot |b(s, \tilde{B}_s^H)| ds \right) (t) \right|^2 \leq C_H \|b\|_{L^\infty}^2 t^{1-2H}.$$

As a result,

$$\int_0^T \left| D_{0+}^{H+\frac{1}{2}} \left( \int_0^\cdot |b(s, \tilde{B}_s^H)| ds \right) (t) \right|^2 dt \leq C_H \|b\|_{L^\infty}^2 \int_0^T t^{1-2H} dt < \infty, \quad P - a.s.$$

since  $H \in (0, 1/2)$ . □

**Lemma 4.3.** *Let  $\tilde{B}_t^H$  be a  $d$ -dimensional fractional Brownian motion with respect to  $(\Omega, \mathfrak{A}, \tilde{P})$ . Then for every  $\mu \in \mathbb{R}$  we have*

$$E \left[ \exp \left\{ \mu \int_0^T \left| K_H^{-1} \left( \int_0^\cdot b(r, \tilde{B}_r^H) dr \right) (s) \right|^2 ds \right\} \right] \leq C_{H,d,\mu,T} (\|b\|_{L^\infty})$$

for some continuous increasing function  $C_{H,d,\mu,T}$  depending only on  $H, d, T$  and  $\mu$ .

In particular,

$$E \left[ \mathcal{E} \left( \int_0^T K_H^{-1} \left( \int_0^\cdot b(r, \tilde{B}_r^H) dr \right)^* (s) dW_s \right)^p \right] \leq C_{H,d,\mu,T} (\|b\|_{L^\infty}),$$

where  $*$  denotes transposition.

*Proof.* Denote by  $\theta_s := K_H^{-1} \left( \int_0^\cdot |b(r, \tilde{B}_r^H)| dr \right) (s)$ . Then using relation (2.5) we have

$$\begin{aligned} |\theta_s| &= |s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} |b(s, \tilde{B}_s^H)| | \\ &= \frac{1}{\Gamma(\frac{1}{2}-H)} s^{H-\frac{1}{2}} \int_0^s (s-r)^{-\frac{1}{2}-H} r^{\frac{1}{2}-H} |b(r, \tilde{B}_r^H)| dr \\ &\leq \|b\|_{L^\infty} \frac{1}{\Gamma(\frac{1}{2}-H)} s^{H-\frac{1}{2}} \int_0^s (s-r)^{-\frac{1}{2}-H} r^{\frac{1}{2}-H} dr \\ &= \|b\|_{L^\infty} \frac{\Gamma(\frac{3}{2}-H)}{\Gamma(1-2H)} s^{\frac{1}{2}-H} \\ &\leq \|b\|_{L^\infty} \frac{\Gamma(\frac{3}{2}-H)}{\Gamma(1-2H)} T^{\frac{1}{2}-H}. \end{aligned}$$

Squaring both sides we have the following estimate

$$|\theta_s|^2 \leq C_H \|b\|_{L^\infty}^2 T^{1-2H} \quad P - a.s., \quad (4.4)$$

where  $C_H := \frac{\Gamma(\frac{3}{2}-H)^2}{\Gamma(1-2H)^2}$ .

Then we get the following estimate

$$E \left[ \exp \left\{ \mu \int_0^T |\theta_s|^2 ds \right\} \right] \leq \exp \left\{ |\mu| C_H T^{2(1-H)} \|b\|_{L^\infty}^2 \right\}.$$

□

By Girsanov's theorem, see Theorem 2.4, the process

$$B_t^H := X_t - x - \int_0^t b(s, X_s) ds, \quad t \in [0, T] \quad (4.5)$$

is a fractional Brownian motion on  $(\Omega, \mathfrak{A}, P)$  with Hurst parameter  $H \in (0, 1/2)$ , where  $\frac{dP}{d\tilde{P}} = \xi_T$ . Hence, because of (4.5), the couple  $(X, B^H)$  is a weak solution of (1.1) on  $(\Omega, \mathfrak{A}, P)$ .

Henceforth, we confine ourselves to the filtered probability space  $(\Omega, \mathfrak{A}, P)$ ,  $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$  which carries the weak solution  $(X, B^H)$  of (1.1).

**Remark 4.4.** As outlined in the scheme above, the main challenge to establish existence of a strong solution is now to show that  $X$  is  $\mathcal{F}$ -adapted. Indeed, in that case  $X_t = F_t(B_t^H)$  for some family of measurable functionals  $F_t$ ,  $t \in [0, T]$  on  $C([0, T]; \mathbb{R}^d)$ , (see e.g. [24] for an explicit form of  $F_t$ ), and for any other stochastic basis  $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P}, \hat{B})$  one gets that  $X_t := F_t(\hat{B}_t)$ ,  $t \in [0, T]$ , is a  $\hat{B}$ -adapted solution to SDE (1.1). But this means exactly the existence of a strong solution to SDE (1.1).

**Remark 4.5.** It is worth to remark that one actually has existence of weak solutions for any  $H \in (0, 1/2)$  and that weak solutions for bounded  $b$  are weakly unique since the estimates from Lemma 4.3 also hold with  $X$  in place of  $\tilde{B}^H$ . For this reason, the main challenge is to show that when  $H$  is small enough such solutions are in fact strong. Then weak uniqueness implies strong uniqueness. See [32].

We now turn to the second step of our procedure.

**Lemma 4.6.** *Let  $\{b_n\}_{n \geq 0} \subset C_c^\infty([0, T] \times \mathbb{R}^d)$  be such that  $\lim_{n \rightarrow \infty} \|b_n - b\|_{L_\infty^1} = 0$  and such that  $|b_n(t, x)| \leq M < \infty$ ,  $n \geq 0$  a.e. for some constant  $M$ . Denote by  $X^n = \{X_t^n, t \in [0, T]\}$  the corresponding solutions of (1.1) if we replace  $b$  by  $b_n$ ,  $n \geq 0$ . Then for every  $t \in [0, T]$  and bounded continuous function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  we have that*

$$\varphi(X_t^n) \xrightarrow{n \rightarrow \infty} E[\varphi(X_t) | \mathcal{F}_t],$$

weakly in  $L^2(\Omega; \mathcal{F}_t)$ .

*Proof.* For a moment let us just, without loss of generality, assume that  $x = 0$ . First we show that

$$\mathcal{E} \left( \int_0^t K_H^{-1} \left( \int_0^\cdot b_n(r, B_r^H) dr \right)^* (s) dW_s \right) \rightarrow \mathcal{E} \left( \int_0^t K_H^{-1} \left( \int_0^\cdot b(r, B_r^H) dr \right)^* (s) dW_s \right) \quad (4.6)$$

in  $L^p(\Omega)$  for all  $p \geq 1$ . To see this, note that

$$K_H^{-1} \left( \int_0^\cdot b_n(r, B_r^H) dr \right) (s) \rightarrow K_H^{-1} \left( \int_0^\cdot b(r, B_r^H) dr \right) (s)$$

in probability for all  $s$ . Indeed, similar computations as in Lemma 4.3 give

$$\begin{aligned} & E \left[ \left| K_H^{-1} \left( \int_0^\cdot b_n(r, B_r^H) dr \right) (s) - K_H^{-1} \left( \int_0^\cdot b(r, B_r^H) dr \right) (s) \right|^2 \right] \\ & \leq \frac{s^{H-1/2}}{\Gamma(\frac{1}{2} - H)} \int_0^s (s-r)^{-1/2-H} r^{1/2-H} E[|b_n(r, B_r^H) - b(r, B_r^H)|] dr \\ & = \frac{s^{H-1/2}}{\Gamma(\frac{1}{2} - H)} \int_0^s (s-r)^{-1/2-H} r^{1/2-H} \int_{\mathbb{R}^d} |b_n(r, y) - b(r, y)| (2\pi r^{2H})^{-d/2} \exp \left\{ -\frac{y^2}{2r^{2H}} \right\} dy dr \\ & \leq C_d \frac{s^{H-1/2}}{\Gamma(\frac{1}{2} - H)} \int_0^s (s-r)^{-1/2-H} r^{1/2-H(d+1)} dr \|b_n - b\|_{L_\infty^1} \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$  since the above integral is finite when  $H < \frac{3}{2(d+1)}$ .

Moreover,  $\{K_H^{-1}(\int_0^\cdot b_n(r, B_r^H) dr)\}_{n \geq 0}$  is bounded in  $L^2([0, t] \times \Omega; \mathbb{R}^d)$ . This is directly seen from (4.4) in Lemma 4.3.

Consequently

$$\int_0^t K_H^{-1} \left( \int_0^\cdot b_n(r, B_r^H) dr \right)^* (s) dW_s \rightarrow \int_0^t K_H^{-1} \left( \int_0^\cdot b(r, B_r^H) dr \right)^* (s) dW_s$$

and

$$\int_0^t \left| K_H^{-1} \left( \int_0^\cdot b_n(r, B_r^H) dr \right) (s) \right|^2 ds \rightarrow \int_0^t \left| K_H^{-1} \left( \int_0^\cdot b(r, B_r^H) dr \right) (s) \right|^2 ds$$

in  $L^2(\Omega)$  since the latter is bounded  $L^p(\Omega)$  for any  $p \geq 1$ , see Lemma 4.3.

Using the estimate  $|e^x - e^y| \leq e^{x+y}|x - y|$ , Hölder's inequality and the bounds in Lemma 4.3 it is clear that (4.6) holds.

Similarly, one also shows that

$$\exp \left\{ \left\langle \alpha, \int_s^t b_n(r, B_r^H) dr \right\rangle \right\} \rightarrow \exp \left\{ \left\langle \alpha, \int_s^t b(r, B_r^H) dr \right\rangle \right\}$$

in  $L^p(\Omega)$  for all  $p \geq 1$ ,  $0 \leq s \leq t \leq T$ ,  $\alpha \in \mathbb{R}^d$ .

To conclude the proof we note that the set

$$\Sigma_t := \left\{ \exp \left\{ \sum_{j=1}^k \langle \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H \rangle \right\} : \{\alpha_j\}_{j=1}^k \subset \mathbb{R}^d, 0 = t_0 < \dots < t_k = t, k \geq 1 \right\}$$

is a total subspace of  $L^2(\Omega, \mathcal{F}_t, P)$  and we may thus restrict ourselves to show the convergence

$$\lim_{n \rightarrow \infty} E[(\varphi(X_t^n) - E[\varphi(X_t)|\mathcal{F}_t]) \xi] = 0$$

for all  $\xi \in \Sigma_t$ . To this end, we notice that  $\varphi$  is of linear growth and hence  $\varphi(B_t^H)$  has all moments. Consequently we have the following convergence

$$\begin{aligned} & E \left[ \varphi(X_t^n) \exp \left\{ \sum_{j=1}^k \langle \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H \rangle \right\} \right] \\ &= E \left[ \varphi(X_t^n) \exp \left\{ \sum_{j=1}^k \langle \alpha_j, X_{t_j}^n - X_{t_{j-1}}^n - \int_{t_{j-1}}^{t_j} b_n(s, X_s^n) ds \rangle \right\} \right] \\ &= E[\varphi(B_t^H) \exp \{ \sum_{j=1}^k \langle \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H - \int_{t_{j-1}}^{t_j} b_n(s, B_s^H) ds \rangle \} \mathcal{E} \left( \int_0^t K_H^{-1} \left( \int_0^\cdot b_n(r, B_r^H) dr \right) (s) dW_s \right)] \\ &\rightarrow E[\varphi(B_t^H) \exp \{ \sum_{j=1}^k \langle \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H - \int_{t_{j-1}}^{t_j} b(s, B_s^H) ds \rangle \} \mathcal{E} \left( \int_0^t K_H^{-1} \left( \int_0^\cdot b(r, B_r^H) dr \right) (s) dW_s \right)] \\ &= E[\varphi(X_t) \exp \{ \sum_{j=1}^k \langle \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H \rangle \}] \\ &= E[E[\varphi(X_t)|\mathcal{F}_t] \exp \{ \sum_{j=1}^k \langle \alpha_j, B_{t_j}^H - B_{t_{j-1}}^H \rangle \}]. \end{aligned}$$

□

We continue to proving the third step of our scheme. This is the most challenging part. The following result is based on a compactness criterion for subsets of  $L^2(\Omega)$  which is summarised in the Appendix.

**Lemma 4.7.** *Let  $\{b_n\}_{n \geq 0} \subset C_c^\infty([0, T] \times \mathbb{R}^d)$  the approximating sequence of  $b$  in the sense of (4.2). Denote by  $X_t^n$  the corresponding solutions of (1.1) if we replace  $b$  by  $b_n$ ,  $n \geq 0$ . Fix  $t \in [0, T]$  then there exists a  $\beta \in (0, 1/2)$  such that*

$$\sup_{n \geq 0} \int_0^t \int_0^t \frac{E[\|D_\theta X_t^n - D_{\theta'} X_t^n\|^2]}{|\theta' - \theta|^{1+2\beta}} d\theta' d\theta \leq \sup_{n \geq 0} C_{H,d,T}(\|b_n\|_{L^\infty}, \|b_n\|_{L^\infty_1}) < \infty$$

and

$$\sup_{n \geq 0} \|D_\theta X_t^n\|_{L^2(\Omega \times [0, T])} \leq \sup_{n \geq 0} C_{H,d,T}(\|b_n\|_{L^\infty}, \|b_n\|_{L^\infty_1}) < \infty \quad (4.7)$$

for some continuous function  $C_{H,d,T} : [0, \infty)^2 \rightarrow [0, \infty)$ .

*Proof.* Fix  $t \in [0, T]$  and take  $\theta, \theta' > 0$  such that  $0 < \theta' < \theta < t$ . Using the chain rule for the Malliavin derivative, see [30, Proposition 1.2.3], we have

$$D_\theta X_t^n = K_H(t, \theta)I_d + \int_\theta^t b'_n(s, X_s^n) D_\theta X_s^n ds$$

$P$ -a.s. for all  $0 \leq \theta \leq t$  where  $b'_n(s, z) = \left( \frac{\partial}{\partial z_j} b_n^{(i)}(s, z) \right)_{i,j=1,\dots,d}$  denotes the Jacobian matrix of  $b$  and  $I_d$  the identity matrix in  $\mathbb{R}^{d \times d}$ . Thus we have

$$\begin{aligned} D_{\theta'} X_t^n - D_\theta X_t^n &= K_H(t, \theta')I_d - K_H(t, \theta)I_d \\ &\quad + \int_{\theta'}^t b'_n(s, X_s^n) D_{\theta'} X_s^n ds - \int_\theta^t b'_n(s, X_s^n) D_\theta X_s^n ds \\ &= K_H(t, \theta')I_d - K_H(t, \theta)I_d \\ &\quad + \int_{\theta'}^\theta b'_n(s, X_s^n) D_{\theta'} X_s^n ds + \int_\theta^t b'_n(s, X_s^n) (D_{\theta'} X_s^n - D_\theta X_s^n) ds \\ &= K_H(t, \theta')I_d - K_H(t, \theta)I_d + D_{\theta'} X_\theta^n - K_H(\theta, \theta')I_d \\ &\quad + \int_\theta^t b'_n(s, X_s^n) (D_{\theta'} X_s^n - D_\theta X_s^n) ds. \end{aligned}$$

Using Picard iteration applied to the above equation we may write

$$\begin{aligned} D_{\theta'} X_t^n - D_\theta X_t^n &= K_H(t, \theta')I_d - K_H(t, \theta)I_d \\ &\quad + \sum_{m=1}^{\infty} \int_{\Delta_{\theta,t}^m} \prod_{j=1}^m b'_n(s_j, X_{s_j}^n) (K_H(s_m, \theta')I_d - K_H(s_m, \theta)I_d) ds_m \cdots ds_1 \\ &\quad + \left( I_d + \sum_{m=1}^{\infty} \int_{\Delta_{\theta,t}^m} \prod_{j=1}^m b'_n(s_j, X_{s_j}^n) ds_m \cdots ds_1 \right) (D_{\theta'} X_\theta^n - K_H(\theta, \theta')I_d). \end{aligned}$$

On the other hand, observe that one may again write

$$D_{\theta'} X_{\theta}^n - K_H(\theta, \theta') I_d = \sum_{m=1}^{\infty} \int_{\Delta_{\theta', \theta}^m} \prod_{j=1}^m b'_n(s_j, X_{s_j}^n) (K_H(s_m, \theta') I_d) ds_m \cdots ds_1.$$

Altogether, we can write

$$D_{\theta'} X_t^n - D_{\theta} X_t^n = I_1(\theta', \theta) + I_2^n(\theta', \theta) + I_3^n(\theta', \theta),$$

where

$$\begin{aligned} I_1(\theta', \theta) &:= K_H(t, \theta') I_d - K_H(t, \theta) I_d \\ I_2^n(\theta', \theta) &:= \sum_{m=1}^{\infty} \int_{\Delta_{\theta', \theta}^m} \prod_{j=1}^m b'_n(s_j, X_{s_j}^n) (K_H(s_m, \theta') I_d - K_H(s_m, \theta) I_d) ds_m \cdots ds_1 \\ I_3^n(\theta', \theta) &:= \left( I_d + \sum_{m=1}^{\infty} \int_{\Delta_{\theta', \theta}^m} \prod_{j=1}^m b'_n(s_j, X_{s_j}^n) ds_m \cdots ds_1 \right) \\ &\quad \times \left( \sum_{m=1}^{\infty} \int_{\Delta_{\theta', \theta}^m} \prod_{j=1}^m b'_n(s_j, X_{s_j}^n) (K_H(s_m, \theta') I_d) ds_m \cdots ds_1 \right). \end{aligned}$$

It follows from Lemma A.4 that

$$\int_0^t \int_0^t \frac{\|I_1(\theta', \theta)\|_{L^2(\Omega)}^2}{|\theta' - \theta|^{1+2\beta}} d\theta d\theta' = \int_0^t \int_0^t \frac{|K_H(t, \theta') - K_H(t, \theta)|^2}{|\theta' - \theta|^{1+2\beta}} d\theta d\theta' < \infty$$

for  $\beta \in (0, 1/2)$ .

Let us continue with the term  $I_2^n(\theta', \theta)$ . Then Girsanov's theorem, Cauchy-Schwarz inequality and Lemma 4.3 imply

$$\begin{aligned} E[\|I_2^n(\theta', \theta)\|^2] &\leq \tilde{C}(\|b_n\|_{L_{\infty}}) E \left[ \left\| \sum_{m=1}^{\infty} \int_{\Delta_{\theta', \theta}^m} \prod_{j=1}^m b'_n(s_j, x + B_{s_j}^H) (K_H(s_m, \theta') I_d - K_H(s_m, \theta) I_d) ds_m \cdots ds_1 \right\|^4 \right]^{1/2}, \end{aligned}$$

where  $\tilde{C} : [0, \infty) \rightarrow [0, \infty)$  is the function from Lemma 4.3. Taking the supremum over  $n$  we have

$$\sup_{n \geq 0} \tilde{C}(\|b_n\|_{L_{\infty}}) =: C_1 < \infty.$$

Let  $\|\cdot\|$  denote the matrix norm in  $\mathbb{R}^{d \times d}$  such that  $\|A\| = \sum_{i,j=1}^d |a_{ij}|$  for a matrix  $A = \{a_{ij}\}_{i,j=1,\dots,d}$ , then taking this matrix norm and expectation we have

$$\begin{aligned} E[\|I_2^n(\theta', \theta)\|^2] &\leq C_1 \left( \sum_{m=1}^{\infty} \sum_{i,j=1}^d \sum_{l_1, \dots, l_{m-1}=1}^d \left\| \int_{\Delta_{\theta', \theta}^m} \frac{\partial}{\partial x_{l_1}} b_n^{(i)}(s_1, x + B_{s_1}^H) \frac{\partial}{\partial x_{l_2}} b_n^{(l_1)}(s_2, x + B_{s_2}^H) \cdots \right. \right. \\ &\quad \left. \left. \cdots \frac{\partial}{\partial x_j} b_n^{(l_{m-1})}(s_m, x + B_{s_m}^H) (K_H(s_m, \theta') - K_H(s_m, \theta)) ds_m \cdots ds_1 \right\|_{L^4(\Omega, \mathbb{R})} \right)^2. \end{aligned}$$



Now look at the expression

$$J_2^n(\theta', \theta) := \int_{\Delta_{\theta,t}^{2m}} \frac{\partial}{\partial x_{l_1}} b_n^{(i)}(s_1, x + B_{s_1}^H) \cdots \frac{\partial}{\partial x_j} b_n^{(l_{m-1})}(s_m, x + B_{s_m}^H) (K_H(s_m, \theta') - K_H(s_m, \theta)) ds. \quad (4.8)$$

Then, shuffling  $J_2^n(\theta', \theta)$  as shown in (2.1), one can write  $(J_2^n(\theta', \theta))^2$  as a sum of at most  $2^{2m}$  summands of length  $2m$  of the form

$$\int_{\Delta_{\theta,t}^{2m}} g_1^n(s_1, B_{s_1}^H) \cdots g_{2m}^n(s_{2m}, B_{s_{2m}}^H) ds_{2m} \cdots ds_1, \quad (4.9)$$

where for each  $l = 1, \dots, 2m$ ,

$$g_l^n(\cdot, B^H) \in \left\{ \frac{\partial}{\partial x_j} b_n^{(i)}(\cdot, x + B^H), \frac{\partial}{\partial x_j} b_n^{(i)}(\cdot, x + B^H) (K_H(\cdot, \theta') - K_H(\cdot, \theta)), i, j = 1, \dots, d \right\}.$$

Repeating this argument once again, we find that  $J_2^n(\theta', \theta)^4$  can be expressed as a sum of, at most,  $2^{8m}$  summands of length  $4m$  of the form

$$\int_{\Delta_{\theta,t}^{4m}} g_1^n(s_1, B_{s_1}^H) \cdots g_{4m}^n(s_{4m}, B_{s_{4m}}^H) ds_{4m} \cdots ds_1, \quad (4.10)$$

where for each  $l = 1, \dots, 4m$ ,

$$g_l^n(\cdot, B^H) \in \left\{ \frac{\partial}{\partial x_j} b_n^{(i)}(\cdot, x + B^H), \frac{\partial}{\partial x_j} b_n^{(i)}(\cdot, x + B^H) (K_H(\cdot, \theta') - K_H(\cdot, \theta)), i, j = 1, \dots, d \right\}.$$

It is important to note that the function  $(K_H(\cdot, \theta') - K_H(\cdot, \theta))$  appears only once in term (4.8) and hence only four times in term (4.10). So there are indices  $j_1, \dots, j_4 \in \{1, \dots, 4m\}$  such that we can write (4.10) as

$$\int_{\Delta_{\theta,t}^{4m}} \left( \prod_{j=1}^{4m} b_j^n(s_j, B_{s_j}^H) \right) \prod_{i=1}^4 (K_H(s_{j_i}, \theta') - K_H(s_{j_i}, \theta)) ds_{4m} \cdots ds_1,$$

where

$$b_l^n(\cdot, B^H) \in \left\{ \frac{\partial}{\partial x_j} b_n^{(i)}(\cdot, x + B^H), i, j = 1, \dots, d \right\}, \quad l = 1, \dots, 4m.$$

The latter enables us to use the estimate from Proposition 3.2 with  $\sum_{j=1}^{4m} \varepsilon_j = 4$  and  $k = 1$  and thus we obtain that

$$E(J_2^n(\theta', \theta))^4 \leq 2^{8m} C^m \|b\|_{L_\infty^1}^{4m} \frac{|\theta' - \theta|^{4\gamma} |t - \theta|^{4m(1-3dH)+4(H-\frac{1}{2}-\gamma)}}{\Gamma(8m(1-3dH) + 1 + 8(H - \frac{1}{2} - \gamma))^{1/2}},$$

whenever  $H < \frac{1}{6d-2}$ .

Altogether, we see that

$$E[\|I_2^n(\theta', \theta)\|^2] \leq \left( \sum_{m=1}^{\infty} d^{m+1} 2^{2m} C^m \frac{\|b_n\|_{L_\infty^1}^m |\theta' - \theta|^\gamma}{\Gamma(8m(1-3dH) + 1 + 8(H - \frac{1}{2} - \gamma))^{1/8}} \right)^2.$$

So we can find a continuous function  $C_{H,d,T} : [0, \infty)^2 \rightarrow [0, \infty)$  such that

$$\sup_{n \geq 0} E [\|I_2^n(\theta', \theta)\|^2] \leq \sup_{n \geq 0} C_{H,d,T}(\|b_n\|_{L^\infty}, \|b_n\|_{L^\infty}) |\theta' - \theta|^\varepsilon$$

for a small enough  $\varepsilon \in (0, 1)$  provided that  $H < \frac{1}{2(3d-1)}$ .

We now turn to the term  $I_3^n(\theta', \theta)$ . Observe that term  $I_3^n(\theta', \theta)$  is the product of two terms, where the first one will simply be bounded uniformly in  $\theta, t \in [0, T]$  under expectation. This can be shown by following meticulously the same steps as we did for  $I_2^n(\theta', \theta)$ .

Again Girsanov's theorem, Cauchy-Schwarz inequality several times and Lemma 4.3 lead to

$$\begin{aligned} E[\|I_3^n(\theta', \theta)\|^2] &\leq \widehat{C}(\|b_n\|_{L^\infty}) \left\| I_d + \sum_{m=1}^{\infty} \int_{\Delta_{\theta', \theta}^m} \prod_{j=1}^m b'_n(s_j, x + B_{s_j}^H) ds_m \cdots ds_1 \right\|_{L^8(\Omega, \mathbb{R}^{d \times d})}^2 \\ &\quad \times \left\| \sum_{m=1}^{\infty} \int_{\Delta_{\theta', \theta}^m} \prod_{j=1}^m b'_n(s_j, x + B_{s_j}^H) K_H(s_m, \theta') ds_m \cdots ds_1 \right\|_{L^4(\Omega, \mathbb{R}^{d \times d})}^2, \end{aligned}$$

where  $\widehat{C} : [0, \infty) \rightarrow [0, \infty)$  denotes the corresponding function obtained from Lemma 4.3 which satisfies

$$\sup_{n \geq 0} \widehat{C}(\|b_n\|_{L^\infty}) =: C_2 < \infty.$$

Again, we have

$$\begin{aligned} E[\|I_3^n(\theta', \theta)\|^2] &\leq C_2 \left( 1 + \sum_{m=1}^{\infty} \sum_{i,j=1}^d \sum_{l_1, \dots, l_{m-1}=1}^d \left\| \int_{\Delta_{\theta', \theta}^m} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x + B_{s_1}^H) \cdots \right. \right. \\ &\quad \left. \left. \cdots \frac{\partial}{\partial x_j} b^{(l_{m-1})}(s_m, x + B_{s_m}^H) ds_m \cdots ds_1 \right\|_{L^8(\Omega, \mathbb{R})} \right)^2 \\ &\quad \times \left( \sum_{m=1}^{\infty} \sum_{i,j=1}^d \sum_{l_1, \dots, l_{m-1}=1}^d \left\| \int_{\Delta_{\theta', \theta}^m} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x + B_{s_1}^H) \cdots \right. \right. \\ &\quad \left. \left. \cdots \frac{\partial}{\partial x_j} b^{(l_{m-1})}(s_m, x + B_{s_m}^H) K_H(s_m, \theta') ds_m \cdots ds_1 \right\|_{L^4(\Omega, \mathbb{R})} \right)^2. \end{aligned}$$

Using exactly the same reasoning as for  $I_2^n(\theta', \theta)$  we see that the first factor can be bounded by some finite constant  $C_3(\|b_n\|_{L^\infty})$  depending on  $H, d, T$  and  $\|b_n\|_{L^\infty}$ , i.e.

$$\begin{aligned} E[\|I_3^n(\theta', \theta)\|^2] &\leq C_3(\|b_n\|_{L^\infty}) \left( \sum_{m=1}^{\infty} \sum_{i,j=1}^d \sum_{l_1, \dots, l_{m-1}=1}^d \left\| \int_{\Delta_{\theta', \theta}^m} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x + B_{s_1}^H) \cdots \right. \right. \\ &\quad \left. \left. \cdots \frac{\partial}{\partial x_j} b^{(l_{m-1})}(s_m, x + B_{s_m}^H) K_H(s_m, \theta') ds_m \cdots ds_1 \right\|_{L^4(\Omega, \mathbb{R})} \right)^2. \end{aligned}$$

As before, look at

$$J_3^n(\theta', \theta) := \int_{\Delta_{\theta', \theta}^{4m}} \frac{\partial}{\partial x_{l_1}} b_n^{(i)}(s_1, x + B_{s_1}^H) \cdots \frac{\partial}{\partial x_j} b_n^{(l_{m-1})}(s_m, x + B_{s_m}^H) K_H(s_m, \theta') ds_m \cdots ds_1. \quad (4.11)$$

We can express  $(J_3(\theta', \theta))^4$  as a sum of, at most,  $2^{8m}$  summands of length  $4m$  of the form

$$\int_{\Delta_{\theta', \theta}^{4m}} g_1^n(s_1, B_{s_1}^H) \cdots g_{4m}^n(s_{4m}, B_{s_{4m}}^H) ds_{4m} \cdots ds_1, \quad (4.12)$$

where for each  $l = 1, \dots, 4m$ ,

$$g_l^n(\cdot, B^H) \in \left\{ \frac{\partial}{\partial x_j} b_n^{(i)}(\cdot, x + B^H), \frac{\partial}{\partial x_j} b_n^{(i)}(\cdot, x + B^H) K_H(\cdot, \theta'), i, j = 1, \dots, d \right\},$$

where the factor  $K_H(\cdot, \theta')$  is repeated four times in the integrand of (4.12). Now we can simply apply Proposition 3.3 with  $k = 1$  and  $\sum_{j=1}^{4m} \varepsilon_j = 4$  in order to get

$$E[(J_3^n(\theta', \theta))^8] \leq 2^{8m} C^m \|b\|_{L_\infty^1}^{4m} \frac{|\theta - \theta'|^{4m(1-3dH)+4(H-\frac{1}{2})}}{\Gamma(8m(1-3dH) + 1 + 8(H - \frac{1}{2}))^{1/2}}.$$

As a result,

$$E[\|I_3^n(\theta', \theta)\|^2] \leq \left( \sum_{m=1}^{\infty} d^{m+1} 2^{2m} C^m \|b\|_{L_\infty^1}^m \frac{|\theta - \theta'|^{m(1-3dH)+H-\frac{1}{2}}}{\Gamma(8m(1-3dH) + 1 + 4(H - \frac{1}{2}))^{1/8}} \right)^2.$$

Hence, there is a continuous function  $C_{H,d,T} : [0, \infty)^2 \rightarrow [0, \infty)$  such that

$$\sup_{n \geq 0} E[\|I_3^n(\theta', \theta)\|^2] \leq \sup_{n \geq 0} C_{H,d,T}(\|b_n\|_{L_\infty}, \|b_n\|_{L_\infty^1}) |\theta' - \theta|^\varepsilon$$

for some  $\varepsilon \in (0, 1)$  small enough provided  $H < \frac{1}{2(3d-1)}$ .

Altogether,

$$\sup_{n \geq 0} \int_0^t \int_0^t \frac{E[\|D_{\theta'} X_t^n - D_\theta X_t^n\|^2]}{|\theta' - \theta|^{1+2\beta}} d\theta' d\theta \leq \sup_{n \geq 0} C_{H,d,T}(\|b_n\|_{L_\infty}, \|b_n\|_{L_\infty^1}) < \infty$$

for  $\beta \in (0, 1/2)$ .

Similar computations show that

$$\sup_{n \geq 0} \|D \cdot X_t^n\|_{L^2(\Omega \times [0, T])} \leq \sup_{n \geq 0} C_{H,d,T}(\|b_n\|_{L_\infty}, \|b_n\|_{L_\infty^1}) < \infty.$$

□

**Corollary 4.8.** *Let  $\{b_n\}_{n \geq 0} \subset C_c^\infty([0, T] \times \mathbb{R}^d)$  the approximating sequence of  $b$  in the sense of (4.2). Denote by  $X_t^n$  the corresponding solutions of (1.1) if we replace  $b$  by  $b_n$ ,  $n \geq 0$ . Then for every  $t \in [0, T]$  and bounded continuous function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  we have*

$$\varphi(X_t^n) \xrightarrow{n \rightarrow \infty} \varphi(E[X_t | \mathcal{F}_t])$$

*strongly in  $L^2(\Omega; \mathcal{F}_t)$ . In addition,  $E[X_t | \mathcal{F}_t]$  is Malliavin differentiable for every  $t \in [0, T]$ .*

*Proof.* This is an immediate consequence of the relatively compactness from Lemma 4.7 and by Lemma 4.6 we can identify the limit as being  $E[X_t|\mathcal{F}_t]$  then the convergence holds for any bounded continuous functions as well. The Malliavin differentiability of  $E[X_t|\mathcal{F}_t]$  is shown by taking  $\varphi = I_d$  and estimate (4.7) together with [30, Proposition 1.2.3].  $\square$

Finally, we can prove the main result of this section.

*Proof of Theorem 4.1.* It remains to prove that  $X_t$  is  $\mathcal{F}_t$ -measurable for every  $t \in [0, T]$  and by Remark 4.4 it then follows that there exists a strong solution in the usual sense that is Malliavin differentiable. Indeed, let  $\varphi$  be a globally Lipschitz continuous function, then by Corollary 4.8 we have, for a subsequence  $n_k, k \geq 0$ , that

$$\varphi(X_t^{n_k}) \rightarrow \varphi(E[X_t|\mathcal{F}_t]), \quad P - a.s.$$

as  $k \rightarrow \infty$ .

On the other hand, by Lemma 4.6 we also have

$$\varphi(X_t^n) \rightarrow E[\varphi(X_t)|\mathcal{F}_t]$$

weakly in  $L^2(\Omega; \mathcal{F}_t)$ . By the uniqueness of the limit we immediately have

$$\varphi(E[X_t|\mathcal{F}_t]) = E[\varphi(X_t)|\mathcal{F}_t], \quad P - a.s.$$

which implies that  $X_t$  is  $\mathcal{F}_t$ -measurable for every  $t \in [0, T]$ .

Finally, to show uniqueness it is enough to show that two given strong solutions are weakly unique, indeed, one can follow the same argument as in [32, Chapter IX, Exercise (1.20)] which asserts that strong existence and uniqueness in law implies pathwise uniqueness. The argument does not rely on the process being a semimartingale. Since our solutions are, by construction, strong and uniqueness in law follows from Novikov's condition from Lemma 4.3 replacing  $B^H$  by  $X$  then pathwise uniqueness follows.  $\square$

## 5. STOCHASTIC FLOWS AND REGULARITY PROPERTIES

Henceforward, we will denote by  $X_t^{s,x}$  the solution to the following SDE driven by a fractional Brownian motion with  $H < 1/2$

$$dX_t^{s,x} = b(t, X_t^{s,x})dt + dB_t^H, \quad s, t \in [0, T], \quad s \leq t, \quad X_s^{s,x} = x \in \mathbb{R}^d. \quad (5.1)$$

We will then assume the hypotheses from Theorem 4.1 on  $b$  and  $H$ . The next result tells us that if  $H = H(k)$  is small enough we may gain regularity on  $x \mapsto X_t^{s,x}$ . In particular, it shows that the strong solution constructed in the former section, in addition to being Malliavin differentiable, is also once weakly differentiable with respect to  $x$  since  $k = 1$ .

**Theorem 5.1.** *Let  $b \in C_c^\infty([0, T] \times \mathbb{R}^d)$ . Fix integers  $p \geq 2$  and  $k \geq 1$ . Then, if  $H < \frac{1}{d(2k+1)}$  we have*

$$\sup_{s,t \in [0,T]} \sup_{x \in \mathbb{R}^d} E \left[ \left\| \frac{\partial^k}{\partial x^k} X_t^{s,x} \right\|^p \right] \leq C_{k,d,H,p,T}(\|b\|_{L^\infty}, \|b\|_{L^1_\infty}),$$

where  $C_{k,d,H,p,T} : [0, \infty)^2 \rightarrow [0, \infty)$  is a continuous function, depending on  $k, d, H, p$  and  $T$ .

*Proof.* Given a  $k$ -times continuously differentiable vector field  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $f(x) = (f^{(1)}(x), \dots, f^{(d)}(x))$ , we have that  $Df : \mathbb{R}^d \rightarrow L(\mathbb{R}^d, \mathbb{R}^d)$  where  $L(\mathbb{R}^d, \mathbb{R}^d)$  denotes the space of linear forms from  $\mathbb{R}^d$  to  $\mathbb{R}^d$  and hence  $Df(x) \in L(\mathbb{R}^d, \mathbb{R}^d)$ . The second derivative is then  $D^2f : \mathbb{R}^d \rightarrow L(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^d))$  so  $D^2f(x) : L(\mathbb{R}^d, L(\mathbb{R}^d, \mathbb{R}^d)) \cong L^2(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$  which is the space of bilinear forms from  $\mathbb{R}^d \times \mathbb{R}^d$  to  $\mathbb{R}^d$ . Further, the  $k$ -th derivative of  $f$  can be seen as an operator  $D^k f : \mathbb{R}^d \rightarrow L(\mathbb{R}^d \times \dots \times \mathbb{R}^d, \mathbb{R}^d)$  and hence  $D^k f(x) \in L(\mathbb{R}^d \times \dots \times \mathbb{R}^d, \mathbb{R}^d)$  being a  $k$ -multilinear form from  $\mathbb{R}^d \times \dots \times \mathbb{R}^d$  to  $\mathbb{R}^d$ . We will further identify the space of  $k$ -multilinear forms with the  $k + 1$ -times tensor product of  $\mathbb{R}^d$ , i.e.

$$L(\mathbb{R}^d \times \dots \times \mathbb{R}^d, \mathbb{R}^d) \cong \otimes_{i=1}^{k+1} \mathbb{R}^d.$$

We endow  $\mathbb{R}^{d \otimes \dots \otimes d}$  with the Kronecker product and denote

$$D^k f(x) = \left\{ \frac{\partial^k}{\partial x_j \partial x_{l_1} \dots \partial x_{l_{k-1}}} f^{(i)}(x) \right\}_{i, l_1, \dots, l_{k-1}, j=1, \dots, d}.$$

Since  $b \in C_c^\infty([0, T] \times \mathbb{R}^d)$ , we know (compare [18]) that the solution of (5.1),  $X_t^{s,x}$  is smooth in the initial value  $x$  and that

$$\frac{\partial}{\partial x} X_t^{s,x} = I_d + \int_s^t Db(u, X_u^{s,x}) \frac{\partial}{\partial x} X_u^{s,x} du.$$

Using Picard's iteration we get

$$\frac{\partial}{\partial x} X_t^{s,x} = I_d + \sum_{m \geq 1} \int_{\Delta_{s,t}^m} Db(u_1, X_{u_1}^{s,x}) \dots Db(u_m, X_{u_m}^{s,x}) du_m \dots du_1. \quad (5.2)$$

Now apply  $\frac{\partial}{\partial x}$  again, then by dominated convergence we have

$$\frac{\partial^2}{\partial x^2} X_t^{s,x} = \sum_{m \geq 1} \int_{\Delta_{s,t}^m} \frac{\partial}{\partial x} [Db(u_1, X_{u_1}^{s,x}) \dots Db(u_m, X_{u_m}^{s,x})] du_m \dots du_1. \quad (5.3)$$

We can expand (5.3) using Leibniz's rule as follows

$$\begin{aligned} \frac{\partial}{\partial x} [Db(u_1, X_{u_1}^{s,x}) \dots Db(u_m, X_{u_m}^{s,x})] \\ = \sum_{r=1}^m Db(u_1, X_{u_1}^{s,x}) \dots D^2b(u_r, X_{u_r}^{s,x}) \frac{\partial}{\partial x} X_{u_r}^{s,x} \dots Db(u_m, X_{u_m}^{s,x}). \end{aligned}$$

Inserting the representation (5.2) for  $DX_t^{s,x}$  in this case we have that

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} X_t^{s,x} &= \sum_{m_1 \geq 1} \int_{\Delta_{s,t}^{m_1}} \sum_{r=1}^{m_1} Db(u_1, X_{u_1}^{s,x}) \cdots D^2 b(u_r, X_{u_r}^{s,x}) \\
&\quad \times \left( I_d + \sum_{m_2 \geq 1} \int_{\Delta_{s,u_r}^{m_2}} Db(v_1, X_{v_1}^{s,x}) \cdots Db(v_{m_2}, X_{v_{m_2}}^{s,x}) dv_{m_2} \cdots dv_1 \right) \\
&\quad \times Db(u_{r+1}, X_{u_{r+1}}^{s,x}) \cdots Db(u_{m_1}, X_{u_{m_1}}^{s,x}) du_{m_1} \cdots du_1. \\
&= \sum_{m_1 \geq 1} \left\{ \int_{\Delta_{s,t}^{m_1}} \sum_{r=1}^{m_1} Db(u_1, X_{u_1}^{s,x}) \cdots D^2 b(u_r, X_{u_r}^{s,x}) Db(u_{r+1}, X_{u_{r+1}}^{s,x}) \cdots Db(u_{m_1}, X_{u_{m_1}}^{s,x}) du_{m_1} \cdots du_1 \right. \\
&\quad + \int_{\Delta_{s,t}^{m_1}} \sum_{r=1}^{m_1} Db(u_1, X_{u_1}^{s,x}) \cdots D^2 b(u_r, X_{u_r}^{s,x}) \\
&\quad \times \left( \sum_{m_2 \geq 1} \int_{\Delta_{s,u_r}^{m_2}} Db(v_1, X_{v_1}^{s,x}) \cdots Db(v_{m_2}, X_{v_{m_2}}^{s,x}) dv_{m_2} \cdots dv_1 \right) \\
&\quad \times Db(u_{r+1}, X_{u_{r+1}}^{s,x}) \cdots Db(u_{m_1}, X_{u_{m_1}}^{s,x}) du_{m_1} \cdots du_1 \left. \right\}.
\end{aligned}$$

We reallocate terms by dominated convergence and respecting the order of matrices

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} X_t^{s,x} &= \sum_{m_1 \geq 1} \sum_{r=1}^{m_1} \int_{\Delta_{s,t}^{m_1}} Db(u_1, X_{u_1}^{s,x}) \cdots D^2 b(u_r, X_{u_r}^{s,x}) \cdots Db(u_{m_1}, X_{u_{m_1}}^{s,x}) du_{m_1} \cdots du_1 \\
&\quad + \sum_{m_1 \geq 1} \sum_{r=1}^{m_1} \sum_{m_2 \geq 1} \int_{\Delta_{s,t}^{m_1}} \int_{\Delta_{s,u_r}^{m_2}} Db(u_1, X_{u_1}^{s,x}) \cdots D^2 b(u_r, X_{u_r}^{s,x}) \\
&\quad \times Db(v_1, X_{v_1}^{s,x}) \cdots Db(v_{m_2}, X_{v_{m_2}}^{s,x}) Db(u_{r+1}, X_{u_{r+1}}^{s,x}) \cdots Db(u_{m_1}, X_{u_{m_1}}^{s,x}) dv_{m_2} \cdots dv_1 du_{m_1} \cdots du_1. \\
&=: I_1 + I_2.
\end{aligned} \tag{5.4}$$

Now, we iterate this scheme, up to step  $k \geq 2$ . We will obtain that  $\frac{\partial^k}{\partial x^k} X_t^{s,x}$  is a sum of  $2^{k-1}$  terms. That is

$$\frac{\partial^k}{\partial x^k} X_t^{s,x} = I_1 + \cdots + I_{2^{k-1}},$$

where each  $I_i$ ,  $i = 1, \dots, 2^{k-1}$  is a sum of iterated integrals over sets of the form  $\Delta_{s,u}^{m_j}$ ,  $s < u < t$ ,  $j = 1, \dots, k$  with integrands having at most one product factor  $D^k b$  and the other factors are of the form  $D^j b$ ,  $j \leq k-1$ .

In order to simplify the reading we introduce some notation. For given indices  $m := (m_1, \dots, m_k)$  and  $r := (r_1, \dots, r_{k-1})$  denote

$$m_j^- := \sum_{i=1}^j m_i \quad \text{and} \quad m_j^+ := \sum_{i=j}^k m_i$$

and

$$\sum_{\substack{m \geq 1 \\ r_l \leq m_l^- \\ l=1, \dots, k-1}} := \sum_{m_1 \geq 1} \sum_{r_1=1}^{m_1} \sum_{m_2 \geq 1} \sum_{r_2=1}^{m_2^-} \cdots \sum_{r_{k-1}=1}^{m_{k-1}^-} \sum_{m_k \geq 1}.$$

Moreover, denote

$$\int_{\Delta^m} \cdot du := \int_{\Delta_{s,t}^{m_1}} \int_{\Delta_{s,u_{r_1}^1}^{m_2}} \cdots \int_{\Delta_{s,u_{r_{k-1}}^{k-1}}^{m_k}} \cdot du$$

where

$$u = (u_{m_k}^k, \dots, u_1^k, \dots, u_{m_1}^1, \dots, u_1^1) \in [0, T]^{m_1^+}.$$

Then, more generally,  $\frac{\partial^k}{\partial x^k} X_t^{s,x} = \sum_{j=0}^{2^{k-1}} I_{2j}$ ,  $k \geq 2$ . We will carry out the computations for  $I_{2^{k-1}}$ , as it can be seen, all terms are treated analogously by choosing  $j = 1, \dots, 2^{k-1}$ . Then  $I_{2^{k-1}}$  will take the following form

$$I_{2^{k-1}}^n = \sum_{\substack{m \geq 1 \\ r_l \leq m_l^- \\ l=1, \dots, k-1}} \int_{\Delta^m} \mathcal{G}_k^X(u) du_1^k \cdots du_{m_k}^k \cdots du_1^1 \cdots du_{m_1}^1,$$

where the integrand  $\mathcal{G}_k^X(u) = \{g_{i,l_1, \dots, l_{k-1}, j}(u)\}_{i,l_1, \dots, l_{k-1}, j=1, \dots, d}$  is an element in  $\mathbb{R}^{d \otimes \dots \otimes d^{k+1}}$  with entries given by sums of at most  $C(d, k)^{m_1 + \dots + m_k}$  terms, which are products of length  $m_1 + \dots + m_k$  with respect to functions belonging to the set

$$g_{i,l_1, \dots, l_{k-1}, j}(u) \in \left\{ \frac{\partial^k}{\partial x_j \partial x_{l_1} \cdots \partial x_{l_{k-1}}} b^{(i)}(u, X_u^{s,x}), i, l_1, \dots, l_{k-1}, j = 1, \dots, d \right\}.$$

Let  $p \in [1, \infty)$  choose  $r, s \in [1, \infty)$  such that  $sp = 2^q$  for some integer  $q$  and  $\frac{1}{r} + \frac{1}{s} = 1$ . Then using Girsanov's theorem in connection with Lemma 4.3 and Hölder's inequality we have for a constant  $C := C(\|b\|_{L^\infty}) > 0$

$$E[\|I_{2^{k-1}}\|^p] \leq CE \left[ \left\| \sum_{\substack{m \geq 1 \\ r_l \leq m_l^- \\ l=1, \dots, k-1}} \int_{\Delta^m} \mathcal{G}_k^{B^H}(u) du_1^k \cdots du_{m_k}^k \cdots du_1^1 \cdots du_{m_1}^1 \right\|^{2^q} \right]^{p/2^q}.$$

Now using the maximum norm on  $\mathbb{R}^{d \otimes \dots \otimes d^{k+1}}$  we get

$$\begin{aligned} & E[\|I_{2^{k-1}}\|^p] \\ & \leq C \left( \sum_{\substack{m \geq 1 \\ r_l \leq m_l^- \\ l=1, \dots, k-1}} \sum_{i \in I} \left\| \int_{\Delta^m} \mathcal{H}_{k,i}^{B^H}(u) du_1^k \cdots du_{m_k}^k \cdots du_1^1 \cdots du_{m_1}^1 \right\|_{L^{2^q}(\Omega, \mathbb{R})} \right)^p, \end{aligned} \quad (5.5)$$

where  $\#I \leq C(d, k)^{m_1 + \dots + m_k}$  and

$$\mathcal{H}_{k,i}^{B^H}(u) := \prod_{l=1}^{m_1^+} h_l(u_l), \quad h_l \in \Lambda, \quad l = 1, \dots, m_1^+,$$

being  $h_l$  elements in the set of functions

$$\Lambda := \left\{ \frac{\partial^k}{\partial x_j \partial x_{l_1} \dots \partial x_{l_{k-1}}} b^{(i)}(u_l, x + B_u^H), \quad i, l_1, \dots, l_{k-1}, j = 1, \dots, d \right\}.$$

Using Lemma 2.1 inductively, one actually shows that

$$\int_{\Delta^m} \mathcal{H}_{k,i}^{B^H}(u) du = \sum_{\sigma \in A_m} \int_{\Delta_{s,t}^{m_1^+}} \prod_{l=1}^{m_1^+} f_{\sigma(l)}(w_l) dw, \quad f_l \in \Lambda, \quad l = 1, \dots, m_1^+, \quad (5.6)$$

for a set of indices  $A_m$  such that  $\#A_m \leq C m_1^+$  for a sufficiently large constant  $C > 0$ .

As a consequence

$$\begin{aligned} & E [\|I_{2^{k-1}}\|^p] \\ & \leq C \left( \sum_{\substack{m \geq 1 \\ r_l \leq m_l^- \\ l=1, \dots, k-1}} \sum_{i \in I} \sum_{\sigma \in A_m} \left\| \int_{\Delta_{s,t}^{m_1^+}} \prod_{l=1}^{m_1^+} f_{\sigma(l)}(w_l) dw \right\|_{L^{2^q}(\Omega, \mathbb{R})} \right)^p, \quad f_l \in \Lambda. \end{aligned} \quad (5.7)$$

Define

$$J := \int_{\Delta_{s,t}^{m_1^+}} \prod_{l=1}^{m_1^+} f_{\sigma(l)}(w) dw, \quad f_l \in \Lambda, \quad l = 1, \dots, m_1^+. \quad (5.8)$$

Then using the same argument as in (4.9) by exploiting the identity in Lemma 2.1 repeatedly, we find that  $J$  can be written to the power 2 as a sum of, at most  $2^{2m_1^+}$  of length  $2m_1^+$  of the form

$$\int_{\Delta_{s,t}^{2m_1^+}} \prod_{l=1}^{2m_1^+} f_{\sigma(l)}(w) dw, \quad f_l \in \Lambda, \quad l = 1, \dots, 2m_1^+.$$

Repeating this argument, we find that we can write  $J^{2^q}$  as a sum of at most  $2^{q2^q m_1^+}$  of length  $2^q m_1^+$  of the form

$$\int_{\Delta_{s,t}^{2^q m_1^+}} \prod_{l=1}^{2^q m_1^+} f_{\sigma(l)}(w) dw, \quad f_l \in \Lambda, \quad l = 1, \dots, 2^q m_1^+.$$

Finally, taking expectation and choosing  $H$  small enough we can apply the estimate from Proposition 3.3 with  $\kappa \equiv 1$  (or  $\varepsilon_j = 0$  for all  $j$ ). Then we can find a constant  $C_T > 0$  such that



$$\begin{aligned}
& (E\|I_{2^{k-1}}\|^p)^{1/p} \leq \\
& \leq C^{1/p} \sum_{\substack{m \geq 1 \\ r_l \leq m_l^- \\ l=1, \dots, k-1}} C(d, k)^{m_1^+} \left( C_T^{2^q m_1^+} 2^{q 2^q m_1^+} \frac{\|b\|_{L_\infty^1}^{2^q m_1^+}}{\Gamma((2^q m_1^+ + 1)(1 - d(2k + 1)H) + 1)^{1/2}} \right)^{1/2^q} \\
& \leq C^{1/p} \sum_{m_1, \dots, m_k \geq 1} (m_1^+)^k C(d, k)^{m_1^+} C_T^{m_1^+} 2^{q m_1^+} \frac{\|b\|_{L_\infty^1}^{m_1^+}}{\Gamma((2^q m_1^+ + 1)(1 - d(2k + 1)H) + 1)^{1/2^{q+1}}} \\
& \leq C^{1/p} \left( \sum_{n \geq 1} C_{d,k,T,p}^n \frac{\|b\|_{L_\infty^1}^n}{\Gamma((2^q n + 1)(1 - d(2k + 1)H) + 1)^{1/2^{q+1}}} \right)^k \\
& < \infty,
\end{aligned}$$

provided  $H < \frac{1}{d(2k+1)}$  for a large enough constant  $C_{d,k,T,p} > 0$  where we used the inequality  $\Gamma(x + y) \geq \Gamma(x)\Gamma(y)$ ,  $x, y \geq 1$  in the third inequality.

As a result,

$$\sup_{s,t \in [0,T]} \sup_{x \in U} E \left[ \left\| \frac{\partial^k}{\partial x^k} X_t^{s,x} \right\|^p \right] \leq C_{k,d,H,p,T} (\|b\|_{L_\infty}, \|b\|_{L_\infty^1})$$

for a continuous function  $C_{k,d,H,p,T} : [0, \infty)^2 \rightarrow [0, \infty)$ , depending on  $k, d, H, p$  and  $T$ .  $\square$

The following is the main result of this section and shows that the fractional Brownian motion  $B^H$  creates a regularising effect on the solution as a function of the initial condition.

**Theorem 5.2.** Assume  $b \in L_\infty^{1,\infty}$ . Let  $U \subset \mathbb{R}^d$  and open and bounded subset and  $X = \{X_t, t \in [0, T]\}$  the solution of (1.1). Then for a small enough Hurst parameter  $H$ , that is  $H < \min \left\{ \frac{1}{2(3d-1)}, \frac{1}{d(2k+1)} \right\}$  it follows that

$$X_t \in \bigcap_{p>1} L^2(\Omega, W^{k,p}(U)).$$

*Proof.* First of all, approximate the irregular drift vector field  $b$  by a sequence of functions  $b_n : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $n \geq 0$  in  $C_c^\infty([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$  in the sense of (4.2). Denote by  $X^{n,x} = \{X_t^{n,x}, t \in [0, T]\}$ , the corresponding solution to (1.1) starting from  $x \in \mathbb{R}^d$  when  $b$  is replaced by  $b_n$ .

Observe that for any test function  $\varphi \in C_0^\infty(U, \mathbb{R}^d)$  and fixed  $t \in [0, T]$  the set of random variables

$$\langle X_t^{n,\cdot}, \varphi \rangle := \int_U \langle X_t^{n,x}, \varphi(x) \rangle_{\mathbb{R}^d} dx, \quad n \geq 0$$

is relatively compact in  $L^2(\Omega)$ . To show this, we use the compactness criterion from Appendix, in Corollary A.3 in terms of the Malliavin derivative. Since the Malliavin derivative is a closed linear operator we have

$$\begin{aligned}
E\left[\int_0^T |D_\theta^j \langle X_t^{n,\cdot}, \varphi \rangle|^2 ds\right] &= \sum_{i=1}^d \left( \int_U E[D_\theta^j X_t^{n,x,(i)}] \varphi_i(x) dx \right)^2 \\
&\leq d \|\varphi\|_{L^2(\mathbb{R}^d, \mathbb{R}^d)}^2 \lambda\{\text{supp}(\varphi)\} \sup_{x \in U} E \left[ \int_0^T \|D_\theta X_t^{n,x}\|^2 ds \right],
\end{aligned}$$

where  $D^j$  denotes the Malliavin derivative in the direction of  $W^{(j)}$ ,  $\lambda$  the Lebesgue measure on  $\mathbb{R}^d$ ,  $\text{supp}(\varphi)$  the support of  $\varphi$  and  $\|\cdot\|$  a matrix norm. Then taking the sum over all  $j = 1, \dots, d$  and using Lemma 4.7 we obtain

$$\sup_{n \geq 0} \|D \cdot \langle X_t^{n,\cdot}, \varphi \rangle\|_{L^2(\Omega \times [0, T])}^2 \leq C \|\varphi\|_{L^2(\mathbb{R}^d, \mathbb{R}^d)}^2 \lambda\{\text{supp}(\varphi)\}.$$

In a similar manner we have

$$\sup_{n \geq 0} \int_0^T \int_0^T \frac{E[\|D_{\theta'} \langle X_t^{n,\cdot}, \varphi \rangle - D_\theta \langle X_t^{n,\cdot}, \varphi \rangle\|^2]}{|\theta' - \theta|^{1+2\beta}} < \infty$$

for  $\beta \in (0, 1/2)$ . Hence  $\langle X_t^{n,\cdot}, \varphi \rangle$ ,  $n \geq 0$  is relatively compact in  $L^2(\Omega)$ . Let us denote by  $Y_t(\varphi)$  its limit after taking (if necessary) a subsequence.

Following exactly the same reasoning as in Lemma 4.6 one can show that

$$\langle X_t^{n,\cdot}, \varphi \rangle \xrightarrow{n \rightarrow \infty} \langle X_t, \varphi \rangle$$

weakly in  $L^2(\Omega)$ . Then by uniqueness of the limit we can establish that

$$Y_t(\varphi) = \langle X_t, \varphi \rangle$$

in  $L^2(\Omega)$ .

Note that there exists a subsequence  $n(j)$  such that  $\langle X_t^{n(j),\cdot}, \varphi \rangle$  converges for every  $\varphi$ , that is,  $n(j)$  is independent of  $\varphi$ .

We have that  $X_t^{n,\cdot}$  is bounded in the Sobolev norm  $L^2(\Omega, W^{k,p}(U))$  for each  $n \geq 0$ . Indeed, by Proposition 5.1 we have

$$\begin{aligned}
\sup_{n \geq 0} \|X_t^{n,\cdot}\|_{L^2(\Omega, W^{k,p}(U))}^2 &= \sup_{n \geq 0} \sum_{i=0}^k E \left[ \left\| \frac{\partial^i}{\partial x^i} X_t^{n,\cdot} \right\|_{L^p(U)}^2 \right] \\
&\leq \sum_{i=0}^k \int_U \sup_{n \geq 0} E \left[ \left\| \frac{\partial^i}{\partial x^i} X_t^{n,x} \right\|^p \right] dx \\
&< \infty
\end{aligned}$$

for a small enough  $H < 1/2$ .

Since  $L^2(\Omega, W^{k,p}(U))$ ,  $p \in (1, \infty)$  is reflexive, by Banach-Alaoglu's theorem we get that the set  $\{X_t^{n,x}\}_{n \geq 0}$  is weakly compact in the  $L^2(\Omega, W^{k,p}(U))$ -topology. Thus, there exists a subsequence  $n(j)$ ,  $j \geq 0$  such that

$$X_t^{n(j),\cdot} \xrightarrow{j \rightarrow \infty} Y \in L^2(\Omega, W^{k,p}(U)).$$

On the other hand, we have proven that  $X_t^{n,x} \rightarrow X_t^x$  strongly in  $L^2(\Omega)$ , so by uniqueness of the limit we can conclude that

$$X_t = Y \in L^2(\Omega, W^{k,p}(U)), \quad P - a.s.$$

Moreover, for all  $A \in \mathcal{F}$  and  $\varphi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$  we have

$$E[1_A \langle X_t, \varphi' \rangle] = \lim_{j \rightarrow \infty} E[1_A \langle X_t^{n(j), \cdot}, \varphi' \rangle] = \lim_{j \rightarrow \infty} -E[1_A \langle \frac{\partial}{\partial x} X_t^{n(j), \cdot}, \varphi \rangle] = -E[1_A \langle Y', \varphi \rangle]$$

and thus

$$\langle X_t, \varphi' \rangle = -\langle Y', \varphi \rangle, \quad P - a.s.$$

□

## APPENDIX A. TECHNICAL RESULTS

The following result which is due to [5, Theorem 1] provides a compactness criterion for subsets of  $L^2(\Omega)$  using Malliavin calculus.

**Theorem A.1.** *Let  $\{(\Omega, \mathcal{A}, P); H\}$  be a Gaussian probability space, that is  $(\Omega, \mathcal{A}, P)$  is a probability space and  $H$  a separable closed subspace of Gaussian random variables of  $L^2(\Omega)$ , which generate the  $\sigma$ -field  $\mathcal{A}$ . Denote by  $\mathbf{D}$  the derivative operator acting on elementary smooth random variables in the sense that*

$$\mathbf{D}(f(h_1, \dots, h_n)) = \sum_{i=1}^n \partial_i f(h_1, \dots, h_n) h_i, \quad h_i \in H, f \in C_b^\infty(\mathbb{R}^n).$$

*Further let  $\mathbb{D}^{1,2}$  be the closure of the family of elementary smooth random variables with respect to the norm*

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|\mathbf{D}F\|_{L^2(\Omega; H)}.$$

*Assume that  $C$  is a self-adjoint compact operator on  $H$  with dense image. Then for any  $c > 0$  the set*

$$\mathcal{G} = \left\{ G \in \mathbb{D}^{1,2} : \|G\|_{L^2(\Omega)} + \|C^{-1} \mathbf{D} G\|_{L^2(\Omega; H)} \leq c \right\}$$

*is relatively compact in  $L^2(\Omega)$ .*

In order to formulate compactness criteria useful for our purposes, we need the following technical result which also can be found in [5].

**Lemma A.2.** *Let  $v_s, s \geq 0$  be the Haar basis of  $L^2([0, T])$ . For any  $0 < \alpha < 1/2$  define the operator  $A_\alpha$  on  $L^2([0, T])$  by*

$$A_\alpha v_s = 2^{k\alpha} v_s, \quad \text{if } s = 2^k + j$$

*for  $k \geq 0, 0 \leq j \leq 2^k$  and*

$$A_\alpha 1 = 1.$$

*Then for all  $\beta$  with  $\alpha < \beta < (1/2)$ , there exists a constant  $c_1$  such that*

$$\|A_\alpha f\| \leq c_1 \left\{ \|f\|_{L^2([0, T])} + \left( \int_0^T \int_0^T \frac{|f(t) - f(t')|^2}{|t - t'|^{1+2\beta}} dt dt' \right)^{1/2} \right\}.$$

A direct consequence of Theorem A.1 and Lemma A.2 is now the following compactness criteria.

**Corollary A.3.** *Let a sequence of  $\mathcal{F}_T$ -measurable random variables  $X_n \in \mathbb{D}^{1,2}$ ,  $n = 1, 2, \dots$ , be such that there exists a constant  $C > 0$  with*

$$\sup_n E[|X_n|^2] \leq C,$$

$$\sup_n E \left[ \|D_t X_n\|_{L^2([0,T])}^2 \right] \leq C$$

and there exists a  $\beta \in (0, 1/2)$  such that

$$\sup_n \int_0^T \int_0^T \frac{E[\|D_t X_n - D_{t'} X_n\|^2]}{|t - t'|^{1+2\beta}} dt dt' < \infty$$

where  $\|\cdot\|$  denotes any matrix norm.

Then the sequence  $X_n$ ,  $n = 1, 2, \dots$ , is relatively compact in  $L^2(\Omega)$ .

For the use of the above result we will need to exploit the following technical results.

**Lemma A.4.** *Let  $H \in (0, 1/2)$  and  $s, t \in [0, T]$ ,  $s < t$ . Then*

$$\int_0^t \int_0^t \frac{|K_H(t, \theta') - K_H(t, \theta)|^2}{|\theta' - \theta|^\gamma} d\theta d\theta' < \infty \quad (\text{A.1})$$

for  $\gamma \in (1, 2)$ .

*Proof.* Write

$$K_H(t, \theta') - K_H(t, \theta) = c_H \left[ f_t(\theta') - f_t(\theta) + \left( H - \frac{1}{2} \right) (g_t(\theta) - g_t(\theta')) \right],$$

where  $f_t(s) := \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}}$  and  $g_t(s) := s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du$ . Hence,

$$\begin{aligned} \int_0^t \int_0^t \frac{|K_H(t, \theta') - K_H(t, \theta)|^2}{|\theta' - \theta|^\gamma} d\theta d\theta' &= \int_0^t \int_0^{\theta'} \frac{(K_H(t, \theta') - K_H(t, \theta))^2}{(\theta' - \theta)^\gamma} d\theta d\theta' \\ &\quad + \int_0^t \int_{\theta'}^t \frac{(K_H(t, \theta) - K_H(t, \theta'))^2}{(\theta - \theta')^\gamma} d\theta d\theta' \\ &=: I_1 + I_2, \end{aligned}$$

where  $I_1$  and  $I_2$  the respective integrals. One can observe that the challenge is to compute either integral. So we will just show the computations for  $I_1$ .

We have

$$\begin{aligned} I_1 &\leq c_H^2 \left[ \int_0^t \int_0^{\theta'} \frac{(f_t(\theta) - f_t(\theta'))^2}{(\theta' - \theta)^\gamma} d\theta d\theta' + \left( H - \frac{1}{2} \right)^2 \int_0^t \int_0^{\theta'} \frac{(g_t(\theta) - g_t(\theta'))^2}{(\theta' - \theta)^\gamma} d\theta d\theta' \right] \\ &=: c_H^2 \left( I_1^1 + \left( H - \frac{1}{2} \right)^2 I_1^2 \right), \end{aligned}$$

where  $I_1^1$  and  $I_1^2$  are the respective integrals above.

Now for  $I_1^2$  one can show that  $g_t(\theta)$  is Hölder-continuous of degree  $1/2$  on  $[0, 1]$ , hence

$$|g_t(\theta) - g_t(\theta')| \leq C|\theta - \theta'|^{1/2}$$

for all  $t \in [0, T]$  and therefore

$$\int_0^t \int_0^{\theta'} \frac{(g_t(\theta) - g_t(\theta'))^2}{(\theta' - \theta)^\gamma} d\theta d\theta' < \infty$$

for  $\gamma \in (1, 2)$ .

For  $I_1^2$  we have

$$I_1^1 \leq \sup_{\theta' \in (0, t)} \left( \frac{t}{\theta'} \right)^{2H-1} \int_0^t \int_0^{\theta'} \frac{((t - \theta)^{H-\frac{1}{2}} - (t - \theta')^{H-\frac{1}{2}})^2}{(\theta' - \theta)^\gamma} d\theta d\theta'.$$

Now apply the change of variables  $v = t - \theta$  and  $wv = \theta' - \theta$  in order to get

$$\begin{aligned} & \int_0^t \int_0^{\theta'} \frac{((t - \theta)^{H-\frac{1}{2}} - (t - \theta')^{H-\frac{1}{2}})^2}{(\theta' - \theta)^\gamma} d\theta d\theta' \\ &= \int_0^t v^{2(H-\frac{1}{2})-\gamma+1} \int_0^1 \frac{(1 - (1 - w)^{H-\frac{1}{2}})^2}{w^\gamma} dw dv. \end{aligned}$$

By using standard techniques one can prove that

$$C := \int_0^1 \frac{(1 - (1 - w)^{H-\frac{1}{2}})^2}{w^\gamma} dw < \infty.$$

for any  $\gamma < 2$ . Finally, integrating with respect to  $v$  we obtain

$$\int_0^t \int_0^{\theta'} \frac{((t - \theta)^{H-\frac{1}{2}} - (t - \theta')^{H-\frac{1}{2}})^2}{(\theta' - \theta)^\gamma} d\theta d\theta' = C \int_0^t v^{2(H-\frac{1}{2})-\gamma+1} dv < \infty$$

provided  $\gamma < 2H + 1$ , i.e.  $\gamma \in (1, 2)$ .  $\square$

**Lemma A.5.** Let  $H \in (0, 1/2)$ ,  $w_j > -1$ ,  $j = 1, \dots, 2m$ ,  $\theta, t \in [0, T]$ ,  $\theta < t$  and  $(\varepsilon_1, \dots, \varepsilon_{2m}) \in \{0, 1\}^{2m}$  be fixed. Then exists a finite constant  $C > 0$  such that

$$\begin{aligned} & \int_{\Delta_{\theta, t}^{2m}} \prod_{j=1}^{2m} (K_H(s_j, \theta) - K_H(s_j, \theta'))^{\varepsilon_j} |s_j - s_{j-1}|^{w_j} ds \\ & \leq C \frac{\prod_{j=1}^{2m} \Gamma(w_j + 1) |\theta' - \theta|^{\gamma \sum_{j=1}^{2m} \varepsilon_j} |t - \theta|^{2m + \sum_{j=1}^{2m} w_j + (H - \frac{1}{2} - \gamma) \sum_{j=1}^{2m} \varepsilon_j}}{\Gamma\left(2m + 1 + \sum_{j=1}^{2m} w_j + (H - \frac{1}{2} - \gamma) \sum_{j=1}^{2m} \varepsilon_j\right)} \end{aligned}$$

for  $\gamma \in (0, H)$ . Observe that if  $\varepsilon_j = 0$  for all  $j = 1, \dots, 2m$  we obtain the classical formula.

*Proof.* First, observe that given exponents  $a, b > -1$  we have

$$\int_\theta^{s_{j+1}} (s_{j+1} - s_j)^a (s_j - \theta)^b ds_j = \frac{\Gamma(a+1) \Gamma(b+1)}{\Gamma(a+b+2)} (s_{j+1} - \theta)^{a+b+1}.$$

Let us start computing

$$\int_\theta^{s_2} (K_H(s_1, \theta) - K_H(s_1, \theta'))^{\varepsilon_1} |s_2 - s_1|^{w_2} |s_1 - \theta|^{w_1} ds_1$$

for  $w_1, w_2 \geq -1$ .

Define the functions  $f_t(s)$  and  $g_t(s)$ ,  $s, t \in [0, T]$ ,  $s \leq t$  as in Lemma A.4. Then for some finite constant  $C_H > 1$

$$(K_H(s_1, \theta) - K_H(s_1, \theta'))^{\varepsilon_1} \leq C_H \left( (f_{s_1}(\theta) - f_{s_1}(\theta'))^{\varepsilon_1} + (g_{s_1}(\theta') - g_{s_1}(\theta))^{\varepsilon_1} \right).$$

For the second term we simply have

$$|g_{s_1}(\theta') - g_{s_1}(\theta)|^{\varepsilon_1} \leq C|\theta' - \theta|^{\varepsilon_1/2}.$$

since  $g_{s_1}$  is Hölder continuous of order  $1/2$  uniformly in  $s_1$ . So

$$\begin{aligned} \int_{\theta}^{s_2} (g_{s_1}(\theta) - g_{s_1}(\theta'))^{\varepsilon_1} |s_2 - s_1|^{w_2} |s_1 - \theta|^{w_1} ds_1 \\ \leq C|\theta - \theta'|^{\varepsilon_1/2} \frac{\Gamma(w_1 + 1) \Gamma(w_2 + 1)}{\Gamma(w_1 + w_2 + 2)} |s_2 - \theta|^{w_1 + w_2 + 1}. \end{aligned}$$

For the term depending on  $f_s$ , as before, observe that

$$\frac{y^{-\alpha} - x^{-\alpha}}{(x - y)^{\gamma}} \leq C y^{-\alpha - \gamma}$$

for every  $0 < y < x < T$  and  $\alpha := (\frac{1}{2} - H) \in (0, 1/2)$  and  $\gamma < \frac{1}{2} - \alpha$ . Hence

$$\begin{aligned} \int_{\theta}^{s_2} (f_{s_1}(\theta) - f_{s_1}(\theta'))^{\varepsilon_1} |s_2 - s_1|^{w_2} |s_1 - \theta|^{w_1} ds_1 \\ \leq C|\theta - \theta'|^{\gamma \varepsilon_1} \int_{\theta}^{s_2} (s_2 - s_1)^{w_2} (s_1 - \theta)^{w_1 - (\alpha + \gamma)\varepsilon_1} ds_1 \\ = C|\theta - \theta'|^{\gamma \varepsilon_1} \frac{\Gamma(w_1 - (\alpha + \gamma)\varepsilon_1 + 1) \Gamma(w_2 + 1)}{\Gamma(w_1 + w_2 - (\alpha + \gamma)\varepsilon_1 + 2)} |s_2 - \theta|^{w_1 + w_2 - (\alpha + \gamma)\varepsilon_1 + 1}. \end{aligned}$$

Observe that

$$\frac{\Gamma(w_1 - (\alpha + \gamma)\varepsilon_1 + 1) \Gamma(w_2 + 1)}{\Gamma(w_1 + w_2 - (\alpha + \gamma)\varepsilon_1 + 2)} \geq \frac{\Gamma(w_1 + 1) \Gamma(w_2 + 1)}{\Gamma(w_1 + w_2 + 2)}$$

for all  $\alpha \in (0, 1/2)$ ,  $\gamma \in (0, \frac{1}{2} - \alpha)$ ,  $w_1, w_2 > -1$  and  $\varepsilon_1 \in \{0, 1\}$ . Altogether we get

$$\begin{aligned} \int_{\theta}^{s_2} (K_H(s_1, \theta) - K_H(s_1, \theta'))^{\varepsilon_1} |s_2 - s_1|^{w_2} |s_1 - \theta|^{w_1} ds_1 \\ \leq C_{H,T} |\theta - \theta'|^{\gamma \varepsilon_1} \frac{\Gamma(w_1 - (\alpha + \gamma)\varepsilon_1 + 1) \Gamma(w_2 + 1)}{\Gamma(w_1 + w_2 - (\alpha + \gamma)\varepsilon_1 + 2)} |s_2 - \theta|^{w_1 + w_2 - (\alpha + \gamma)\varepsilon_1 + 1}. \end{aligned}$$

Integrating iteratively we obtain the desired formula.  $\square$

Finally, we give a similar estimate which is used in Lemma 4.7.

**Lemma A.6.** *Let  $H \in (0, 1/2)$ ,  $w_j > -1$ ,  $j = 1, \dots, 2m$ ,  $\theta, t \in [0, T]$ ,  $\theta < t$  and  $(\varepsilon_1, \dots, \varepsilon_{2m}) \in \{0, 1\}^{2m}$  be fixed. Then exists a finite constant  $C > 0$  such that*

$$\begin{aligned} \int_{\Delta_{\theta,t}^{2m}} \prod_{j=1}^{2m} (K_H(s_j, \theta))^{\varepsilon_j} |s_j - s_{j-1}|^{w_j} ds \\ \leq C \frac{\prod_{j=1}^{2m} \Gamma(w_j + 1) |t - \theta|^{2m + \sum_{j=1}^{2m} w_j + (H - \frac{1}{2}) \sum_{j=1}^{2m} \varepsilon_j}}{\Gamma\left(2m + 1 + \sum_{j=1}^{2m} w_j + (H - \frac{1}{2}) \sum_{j=1}^{2m} \varepsilon_j\right)}. \end{aligned}$$

Observe that if  $\varepsilon_j = 0$  for all  $j = 1, \dots, 2m$  we obtain the classical formula.

*Proof.* Let us start computing

$$\int_{\theta}^{s_2} (K_H(s_1, \theta))^{\varepsilon_1} |s_2 - s_1|^{w_2} |s_1 - \theta|^{w_1} ds_1$$

for  $w_1, w_2 \geq -1$ .

Define the functions  $f_t(s)$  and  $g_t(s)$ ,  $s, t \in [0, T]$ ,  $s \leq t$  as in the proof of Lemma A.4. Then for some finite constant  $C_{H,T} > 0$

$$|K_H(s_1, \theta)|^{\varepsilon_1} ds \leq C_{H,T} (|f_{s_1}(\theta)|^{\varepsilon_1} + |g_{s_1}(\theta)|^{\varepsilon_1}) \leq C_{H,T} (|f_{s_1}(\theta)|^{\varepsilon_1} + |\theta|^{\varepsilon_1/2}).$$

Then we have

$$\begin{aligned} \int_{\theta}^{s_2} (K_H(s_1, \theta))^{\varepsilon_1} |s_2 - s_1|^{w_2} |s_1 - \theta|^{w_1} ds_1 \\ \leq C_{H,T} \frac{\Gamma(w_1 + (H - \frac{1}{2})\varepsilon_1 + 1) \Gamma(w_2 + 1)}{\Gamma(w_1 + w_2 + (H - \frac{1}{2})\varepsilon_1 + 2)} |s_2 - \theta|^{w_1 + (H - \frac{1}{2})\varepsilon_1 + 1}. \end{aligned}$$

Integrating iteratively one obtains the desired estimate.  $\square$

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